Learning and teaching complex cultural knowledge calls for meaningful participation in different kinds of symbolic practices, which in turn are supported by a wide range of external representations, as gestures, oral language, graphic representations, writing and many other systems designed to account for properties and relations on some 2- or 3-dimensional objects. Children start their apprenticeship of these symbolic practices very early in life. But being able to understand and use them in fluid and flexible ways poses serious challenges for learners and teachers across educational levels, from kindergarten to university.

This book is intended as a step in the path towards a better understanding of the dynamic relations between different symbolic practices and the acquisition of knowledge in various learning domains, settings and levels. Researchers from almost twenty institutions in three different continents present first hand research in this emerging area of study and reflect on the particular ways and processes whereby participation in symbolic practices based on a diversity of external representations promotes learning in specific fields of knowledge.

The book will be useful for persons interested in education, as well as cognitive psychologists, linguists and those concerned by the generation, appropriation, transmission and communication of knowledge.
Representational Systems and Practices as Learning Tools
Representational Systems and Practices as Learning Tools

Christopher Andersen  
*The Ohio State University, Columbus, USA*

Nora Scheuer  
*CONICET, Argentina*

María del Puy Pérez Echeverría  
*Universidad Autónoma de Madrid, Spain*

Eva V. Teubal  
*David Yellin Teacher’s College, Jerusalem, Israel*
A C.I.P. record for this book is available from the Library of Congress.


Published by: Sense Publishers,
P.O. Box 21858, 3001 AW
3001 AW Rotterdam
Rotterdam, The Netherlands
http://www.sensepublishers.com
Printed on acid-free paper

No part of this work may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission from the Publisher, with the exception of any material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work.
We dedicate this collective book to the memory of Giyoo Hatano, who inspired many of those who contributed to this project. Were it not for his untimely passing away, this volume would have included a chapter by him. We deeply miss him.
# TABLE OF CONTENTS

Acknowledgments ix

1. External Representations as Learning Tools: An Introduction  1
   *Maria del Puy Pérez Echeverría and Nora Scheuer*

2. From One to Two: Observing One Child’s Early Mathematical Steps 19
   *Nora Scheuer and Anne Sinclair*

3. Young Children’s Developing Ability to Produce Notations in Different Domains — Drawing, Writing, and Numerical 39
   *Esti Klein, Eva Teubal and Anat Ninio*

4. Space-Time Representations in Young Children: Thinking Through Gestures in Motion Experiments 59
   *Ornella Robutti*

5. Learning Language Through Preschool Science 77
   *Lucia French and Shira Peterson*

6. Children’s Semantic Representations of a Science Term 93
   *Rachel Best, Julie Dockrell and Nick Braisby*

7. Children’s Representations in Modelling Scientific Knowledge Construction 109
   *Andrés Acher and Maria Arcà*

8. Tables as Cognitive Tools in Primary Education 133
   *Eduardo Martí*

9. Does Drawing Contribute to Learning to Write? Children Think it Does 149
   *Nora Scheuer, Montserrat de La Cruz, Juan Ignacio Pozo and Maria Faustina Huarte*

    *Merce Garcia-Mila, Christopher Andersen and Nubia E. Rojo*

11. The Impact of Labeling on Adults’ and Children’s Ability to Use Geometrical Definitions 187
    *Eva Teubal, Ainat Guberman and Jeanne Albert*
# TABLE OF CONTENTS

12. Graphicacy: University Students’ Skills in Translating Information  
*Maria del Puy Pérez Echeverría, Yolanda Postigo and Ana Pecharromán*  
209

13. What Does “In the Infinite” Mean?: The Difficulties with Dealing with the Representation of the “Infinite” in a Teaching Sequence on Optics  
*Eduardo F. Mortimer and Christian Baty*  
225

14. Representing Organic Molecules: The Use of Chemical Languages by University Students  
*Juan Ignacio Pozo and Maria Gabriela Lorenzo*  
243

15. Writer Development in the Sciences: Expressing New Meanings in Research  
*Ann Montemayor-Borsinger*  
267

16. A Reading of the Volume from the Perspective of Symbol-Use  
*Ricardo Nemirovsky*  
281

17. External Representations Critical to Human Intelligence: Reflections on the Volume  
*Katherine Nelson*  
297

Final Words  
315

Author Index  
317
ACKNOWLEDGMENTS

Through their comments, suggestions, and criticisms, these external reviewers helped improve the chapters of this book. To each of them, we extend our heartfelt thanks for their thoughtful effort.

Mónica Alvarado
Abraham Arcavi
Beatriz Barquero
Diane Beals
Marcelo Borba
Bárbara Brizuela
Onno De Jong
Zoltan Dienes
Julio C. Jiménez
Anne Grobet
Natalia Ignatieva
Maria Pilar Jiménez-Alexandre
Richard Lehrer
Linda McGuigan
Maria Alessandra Mariotti
Sindhu Mathai
Eun Jung Park
Luis Radford
Meredith Rowe
Elizabeth Simmonds
Carol Smith
Liliana Tolchinsky
Ernst von Glasersfeld
Zhenlin Wang

In addition, we wish to thank Lawrence Anderson for his assistance in the preparation of this volume.
1. EXTERNAL REPRESENTATIONS AS LEARNING TOOLS

An Introduction

Nothing has really happened until it’s been described. So you must write many letters to your family and friends, and keep a diary. (Virginia Woolf, quoted by S. P. Rosenbaum, 1995, p. 334)

Literature, philosophy, history, psychology, biology, and neuroscience, as well as our own everyday experience, show that there are very many ways in which something can happen to somebody. There is a myriad of apparently imperceptible events and states that leave their track in the structure and dispositions of mindful beings. However, a deep truth seems to lie in Virginia Woolf’s words. At a certain level, one gets the impression that things really, or only, happen to us when we are able to express them through some kind of public, overt, palatable means — or recognize them in expressions provided by others. When we capture events, sensations, or intuitions within ‘descriptions’, we prevent them (even if partially and temporarily) from vanishing away as they become indistinguishably melded with past and forthcoming, fleeting configurations. Descriptions cut off an event from the continuous flow of experience and maintain it in the focus of attention for a period long enough to highlight it to self and, in social contexts, to others. Descriptions are symbolic representations; they refer to aspects of objects or situations according to a particular perspective. Being able to ‘describe’ an experience requires and enhances several high-order processes that humans participating in complex social and cultural environments regularly carry out: communication (Bruner, 1990; Nelson, 1996), memory externalization (Donald, 1991), awareness and consciousness (Dehaene & Naccaché, 2001), learning and transmission (Tomasello, Kruger & Ratner, 1993); knowledge explicitation (Dienes & Perner, 1999; Karmiloff-Smith, 1992), adopting and sharing models or perspectives (Olson, 1994; Tomasello, 1999; Tomasello, Carpenter, Call, Behne & Moll, 2005), extension of a sense of self (Damasio, 1999; Nelson, 1993).

While Virginia Woolf was referring to the generative power of writing — writing as creating experience as it renders it explicit, rather than as transposing previously experienced contents directly onto paper, we can spread out this idea to many other overt kinds of representations. According to Donald’s (1991) speculative reconstruction of the evolution of culture and human cognition (extensively reviewed by Katherine Nelson in Chapter 17), well before writing systems were developed, mimetic, linguistic, and visuospatial systems of representation made it possible to
generate retrievable memories of various kinds (motor, lexical, narrative, pictorial), distil and adjust them, build up a shared semantic environment, communicate social histories that contribute to consolidating and extending cultural identities, and draw attention to a diversity of relations of similarity, contiguity, and proximity, as well as functional, causal, and hierarchical. Although in long-term human evolution, these different systems have emerged successively, nowadays they all continue to operate in close interaction with each other and support the modern ‘hybrid mind’ (Donald, 2001).

Mimesis (or the use of the whole body or some of its parts as a representational device), oral language, graphic images, writing, and other notational systems share a noteworthy property: they are means of representation that generate some kind of intentional and socially overt product, be it a greeting performed by means of a rather standardized body movement or gesture, an utterance, a drawing, a record of quantity, a signature, or a linearly organized text. Just as we pointed out for the particular case of ‘descriptions’, the diverse material embodiments of representational processes offer a foundation for social communication; cultural production, preservation, transmission, and change; and, at a cognitive level, for knowledge acquisition, refinement, and change, and even for the construction of self. The impact this material embodiment produces both at a sociocultural and a cognitive level is such that it justifies assembling these diverse representational means under the common term external representations.

The aim of this volume is to highlight and reconsider the importance of a wide — but not exhaustive — range of external representations in learning and teaching. The effects of coming to learn these semiotic tools go well beyond becoming able to access an impressive volume of millenary cultural knowledge. As shown by Vygotsky (1978) and by many other more recent studies (Bereiter & Scardamalia, 1987; Martí, 2003; Nelson, 1996; Nunes & Bryant, 1996; Olson, 1994; Pozo, 2001, 2003), cultural symbol systems endow learners with action patterns to operate in standardized or challenging situations and with models to account for the physical, social, and mental world. External representations are not only avenues to knowledge; they are forms of knowing. It seems to us that exploiting the cultural and cognitive potential provided by the variety of external representations ‘alive’ in contemporary societies poses a theoretical and educational challenge that deserves study and reflection. This is the main concern that gave rise to this volume and weaves its contents together.

In the following section, we briefly analyse some of the features that signal relevant distinctions among four major kinds of external representations: bodily and gestural representations, oral language, notations based on relatively relaxed combination rules (as drawing, maps, and some kinds of graphs), and more strict notational systems (as alphabetically based writing, the written number system, algebraic systems, among others). However, the distinction between ‘strict’ and ‘relaxed’ rules is better understood in terms of a continuum than of an opposition. In accord with the contents of this volume, we leave many other significant kinds of external representations aside: deaf sign language, reading and writing Braille system, video and film, written records of choreographies, and the combined use of several external
representations in computerized environments, to mention only a few. In the second section, we justify our choice of the adjective ‘external’ for this diversity of representations. In the third section, we elaborate on the psychological model of progressive knowledge explicitation proposed by Dienes and Perner (1999) as an appealing perspective to think about the roles of external representations in learning and teaching. External representations not only contribute to make knowledge content more visible to learners, but also to expand their consciousness of how they (and others) relate to the content of knowledge (knowing attitude, for Dienes & Perner) and change (knowing self) as they learn. Different modes in which external representations take part in the explicitation of knowledge content, attitude, and self are illustrated by considering the findings of the 14 studies collected in this volume, in reference to a wide range of learning domains, contexts, and agents.

FEATURES OF EXTERNAL REPRESENTATIONS

In our examination of the main features of bodily or gesture representations, oral language, and notational systems, we rely on Donald’s analysis (1991) — which is also the starting point for Katherine Nelson’s penetrating reflections in Chapter 17. We approach these features in broad terms, almost without accounting for differences within kinds of representation (for the analysis of different body-based counting systems, see Saxe, 1991; for numerical notation, Ifrah, 1985; for writing systems, Coulmas, 1996).

However, we depart from Donald as regards two terminological choices. Instead of ‘mimetic’, we speak of ‘body or gesture’ representations, in order to specify the physical materialization of these representations, without assuming that they are perforce imitative (as indicated by some gesture representations shown in Chapters 2 and 4 in the present volume). Besides, in accord with research in developmental psychology (Karmiloff-Smith, 1992; Martí, 2003) we prefer ‘notational systems’ to ‘visuospatial systems’. Notations are graphic marks produced by means of some intermediate instrument (though on occasion body parts are used) on some kind of surface (skin, sand, walls, floors, clay, paper, cloth, screens). Many notations ‘live’ long after they have been produced, and thus may be retrieved over and over again by their makers and others (as the successive draft versions of an effortful text), or may be even found at distant times and places (well-known examples are the Hammurabi code, Inca kipus, or the Nazca lines). The expression ‘notational systems’ indicates cases in which notations form part of a system, so that the meaning of a particular notation is derived from its relation with other notations in the system. We establish a distinction between a) notational systems that are principally based on rather relaxed grammars and analogical resources, and b) systems that are based on sets of components and on combination and composition rules that are defined in a more strict manner. (See Nelson Goodman (1976) for a different approach – while our distinction may appear similar to his dense and articulated symbols respectively, in Goodman’s theory of symbols it is only articulated symbols that form systems). Cases of the former are pictorial representation, and to a certain extent, some kinds of diagrams, schemes, and maps (which often integrate writing
and mathematical notation in diverse ways). Cases of the latter are the diverse systems to note mathematical entities, procedures, and relations; to record the pitch, duration, and intensity of sounds, or the *tempo*, rhythm, and structure of a musical piece; a variety of ideographic, syllabic, phonetic, and mixed writing systems.

Bodily and gesture representations, oral language, relatively ‘relaxed’ and considerably strict notational systems, while all being external representations, present relevant differences that in turn lead to differential social uses and cognitive potentials (see contents in oval in Figure 1 below). In Figure 1 we have chosen to arrange these four major kinds of external representations according to two spatial dimensions. The vertical axis indicates an opposition between the dimensions (temporal vs. spatial, see Martí, 2003, 2005) over which representations become externalized, whereas the horizontal axis marks the gradient of structural definition involved. (Of course, there are many exceptions to these distinctions, but in general terms they highlight well-documented tendencies).

To begin with (see first difference listed in Figure 1), the physical gap between contexts and agents of production on the one hand, and of reception on the other,
EXTERNAL REPRESENTATIONS AS LEARNING TOOLS

widens as we move from gesture representations to oral language, and next to notations. In the case of body-based representations it is necessary that producer and receptor concur closely in a particular space-time for communication to occur (of course, when video recording devices are not employed, or before they were developed). In the case of oral language, instead, communication can take place when participants are tens of meters apart and even in the dark (Donald, 1993). Since notations may be displayed on relatively long-lasting bidimensional surfaces, they afford being perceived long after they have been produced and in absence of the maker. Certainly, notational technologies have played and continue to play a central part here — clay tablets, papyri, printed sheets sewn in hard covered books, or files backed up in the cyberspace afford retrieval upon ever more extended temporal and geographical distance. Paradoxically, as the physical gap grows between contexts/agents of production and of reception, social participation in production and social access to the resulting products rise. New forms of knowledge building, coauthorship, and intertextuality come about, facilitated by diverse communication and editing tools — graphic, textual, audiovisual (Scardamalia & Bereiter, 2003).

The distance across the gap between contexts of production and reception is closely related to the differential duration of material existence of products corresponding to different kinds of external representations. The physical materialization over time attained by some kinds of external representations contributes to their ‘objectification’ (Olson, 1994) or ‘reification’ (Sfard, 2002). This is particularly evident for notations, which may be recovered long after original production, depending on the surface employed and subsequent manipulation and care, among other factors. But also body representations may be held still for a moment, and hence offer a momentarily stable and relatively detached support for thought. A particularly illustrative example is the Oksapmin’s counting system, which allows these people to count up to 27 by relying on an ordered sequence of body parts in absence of any writing system, as documented by Saxe (1991). Another example of this is young children’s need to display finger patterns in order to perform calculations with quantities up to around 20 — though the familiarity with complex finger-counting systems allows extending this limit much further (see Fuson, 1991). As learning proceeds and children come to internalize addition procedures, they may no longer recur to enact the addends, the addition operation, or its result for such numerical intervals. But under the request to perform calculations with larger quantities or involving many steps, older children and even adults may need to display their thought on paper for it to proceed. In turn, the ‘objective’ quality of external representations, particularly of notations, enhances a repertoire of metacognitive skills (see for instance, Flavell, 1987) to revise products as well as to tune and monitor production procedures, in order to accomplish higher semantic and syntactic standards and pragmatic efficiency.

External representations endow persons and cultures with an extraordinary amount and variety of partly ‘prepackaged’ symbolic tools, even if deliberate learning and/or teaching are generally required to become capable of using them in autoregulated, flexible, and effective ways. “… we carry around tens of thousands, and in the case of some multilinguals, hundreds of thousands of words; most other species, from bees to
Great Apes, seem to be limited to a few dozen expressions at the most in the wild” (Donald, 1993, p. 744). This vast representational repertoire brings about considerable referential precision, which tends to increase when we move from bodily representations to oral language, and still further when we encounter strictly organized notational systems. However, it is rather difficult to situate pictorial representations according to this criterion. Is a picture worth more or less than one thousand words? The answer depends on the particular text or picture, as well as on the information to be conveyed and the communicative purpose. (For a discussion of the relative contributions of graphic and textual expressions, see Postigo & Pozo, 2004.) The amount and variety of external representations social groups count on set the scene for an increasingly complex, dense, and pervasive shared semantic environment. Appealing to a reversal device in his dystopian novel Nineteen Eighty-Four, George Orwell offers a compelling clue of the social and cognitive power of lexically rich representational systems. As a mechanism directed at annihilating social links, critical thought, and personal identity in this highly controlled totalitarian society, every successive edition of the official dictionary is shorter. Syme, who is deeply involved with the mission of reducing ‘Oldspeak’ into ‘Newspeak’, expresses the point crudely:

You think, I dare say, that our chief job is inventing new words. But not a bit of it! We’re destroying words – scores of them, hundreds of them, every day. We’re cutting the language down to the bone. (Orwell, 2004, p. 66)

Don’t you see that the whole aim of Newspeak is to narrow the range of thought? In the end we shall make thoughtcrime literally impossible, because there will be no words in which to express it. Every concept that can ever be needed, will be expressed by exactly one word, with its meaning rigidly defined and all its subsidiary meanings rubbed out and forgotten. (...) Every year fewer and fewer words, and the range of consciousness always a little smaller. (Orwell, 2004, p. 68)

Most external representations are organized in more or less structured systems. In these cases, the meanings attributed to particular representational exemplars stem from their relations to other representations within the system or within the notational context. We can find a paradigmatic example in the hindu-arabic system for the notation of natural numbers. The meaning of 0 — a graphic mark that paradoxically signals the absence of quantity in a system built to express quantities — may only be understood in relation to the principles that organize the system: base and position, so that a 0 on its own indicates the absolute absence of quantity in counting contexts, but 0 in the numeral 10 indicates the absence of ‘free’ or ungrouped units, whereas in 101 it signals the absence of ‘free’ groups of ten. Turning to the domain of drawing, we find that the typical pictorial representation of a house with its tile roof and smoking chimney is more closely related to the conventional pictorial representation of the category house than to the 3-dimensional urban houses the drawing child or adult may have inhabited or visited (see Sinclair, 1988, p. 15, for this kind of analysis).

The organized nature of external representation systems provides a structuring envelope
EXTERNAL REPRESENTATIONS AS LEARNING TOOLS

for personal and social experience. In the very act of expressing a content according to a preestablished, standardized arrangement, such content is laden (even though unnoticeably) with new and different meanings. Again, the complexity of the organizing grammars and the extent to which they are made explicit to users, teachers, and learners vary according to kinds of external representations. Normally, it is within highly structured notational systems (as writing or as numerical, chemical, or musical notation) that grammatical rules present higher levels of explicitation, together with an increase in simplicity and hence in economy. Organizing grammars privilege establishing certain relations and adopting certain perspectives, above other alternatives. According to Olson’s compelling analysis (1994), visually supported representations as pictures, maps, different writing systems, as well as the related writing and reading practices, provide groups and individuals with deeply entrenched models of the physical, cultural, and mental world. External representations function as Piagetian structuring structures (see for instance Piaget, 1963) — despite the fact that he used this expression to denote mental constructs.

Though for the sake of analysis we have treated the different kinds of external representations as separate from each other, they continually coalesce in everyday activities. In this multirepresentational environment, persons have the opportunity to use a very rich representational gear, switch among and combine representational alternatives. The use of representational alternatives usually opens the range of possible ways to approach a situation, allows relating to it from different angles and visualising new combinations of operations and new solutions. Hence, external representations, especially when used in combination, seem to expand problem solving capacity and conceptual perspectivism (Clark, 1997; Gagatsis & Shiakalli, 2004).

A PROCESS VIEW OF EXTERNAL REPRESENTATIONS

Several authors have resorted, in one way or another, to the term ‘external’ representations (Martí, 2003; Martí & Pozo, 2000) or ‘external’ representation of memory (Donald, 1991) in order to distinguish representational practices and processes that result in some kind of intentional and public manifestation (gestures, spoken or written language, pictures, photographs, diagrams, graphs, 3-dimensional objects) from others that can occur solely in a private scene (mental images, thoughts, feelings; see Damasio, 1999). As it happens with almost every lexical choice for a theoretical construct, its adoption brings about some shade of discomfort. In this case, the phantom of dualism seems to show up: are we by chance establishing an absolute frontier between outer and inner worlds? (See Ricardo Nemirovsky’s criticism of dualism in Chapter 16.) This is not our intention. Rather, we consider ‘external’ and ‘internal’ as simplified ways of referring to subtle and multifaceted processes. It seems to us that, for the time being, such terms offer the advantage of clearly signalling the noteworthy difference between a) instances of representational patterns and processes that are partly displayed in overt, visible, audible, or palpable scenarios, and b) instances of representational patterns and processes that can occur exclusively within private scenarios, so that their material (neurobiological) correlates can only be observed or recorded with the help of complex technologies (Dehaene &
Naccaché, 2001). This is a crucial distinction, given that getting in touch, producing, using, and interpreting external representations support and extend human ways of acting and constructing meaning.

However, we should keep in mind that ‘external’ representations are externalizations of mental processes; and most mental representations (especially those that comprise certain elaboration) involve internalization processes. Just as it occurs with many other cognitive and cultural processes, the use of an adjective (‘external’, ‘internal’) may lead to conceive of them as fixed. Stewart (1995) has discussed such semantic shifts in relation to mathematics, by bringing to light the subtle ways whereby numbers, i.e., entities that ontologically correspond to conceptual processes but are referred to by nouns, at a lay level of understanding slip into the ontological terrain of ‘objects’. Getting back to our topic, what we are choosing to call ‘external representations’ stem from a complex cultural and personal history, and thus not only involve a product-like dimension, but also a process-like one (see Tolchinsky, 2006, for the case of writing). External representations often are the effortful result of successive collective and individual attempts to make intuitions, images, emotions, and experiences visible, audible, palpable, or available to self and others. At the same time (or at subsequent cultural and personal times), externalization processes and their resulting products provoke a lasting impact on the ways of understanding. In sum, internal representations are not univocally linked to fixed corresponding external representations. Rather, external and internal representations form part of a tightly interwoven tissue, as the collection of chapters in this volume show in many ways.

Several contributions in this volume stress the challenges children and adults find in their way to internalize diverse cultural external representations: the apparently simple finger pattern for a collection of 3 or the number word ‘two’ (Chapter 2 by Scheuer & Sinclair); the differential social uses of drawing, numerical notation, and writing, according to the kind of information to be conveyed (Chapter 3, by Klein, Teubal, & Ninio); graphic representations and 3-dimensional models of the solar system (Chapter 6, by Best, Dockrell, & Braisby); double entry tables (Chapter 8, by Marti), schemas and diagrams (Chapter 13, by Mortimer & Buty), Cartesian graphs (Chapter 12, by Pérez Echeverría, Postigo, & Pecharromán); geometric and chemical nomenclature (Chapter 11, by Teubal, Guberman, & Albert; Chapter 14, by Pozo & Lorenzo); algebraic formulation (Chapter 12) and ‘abstract representations’ (Sperber, 1996) lacking direct empirical reference, such as the concept of infinite (Chapter 13).

The processes learners go through and the difficulties they encounter when they attempt to express their understandings in publicly accessible conventional representations are also analysed throughout the volume. The understandings to be some how expressed deal with a varied range of contents: very small quantities of present or absent objects (Chapter 2); dates, ages, and names (Chapter 3); the kind and amount of personal belongings or of items in a shopping list (Chapters 8 and 3); geometrical shapes (Chapter 11); the image of oneself (Chapter 9, by Scheuer, de la Cruz, Pozo, & Huarte). Other contents regard processes studied in biology (Chapter 10, García-Mila, Andersen, & Rojo), chemistry (Chapter 14), physics
EXTERNAL REPRESENTATIONS AS LEARNING TOOLS

(Chapter 4, by Robutti; Chapter 5, by French & Peterson; Chapter 7, by Acher & Arcà; Chapter 13), and astronomy (Chapter 6), in relation to ongoing or very close experience (Chapters 4, 5, and 7), recalled experience (Chapters 6 and 10), or to a theoretic space (Chapters 13 and 14). Attention is also drawn to translating or trans-coding information from one kind of external representation to another (Chapters 12 and 14). Just as it happens with translations from one language to another, such passage unavoidably brings about the loss of some shades of meaning (traduttore, traditore! according to the Italian expression), at the time it may favour new ins-sights and redescriptions (Karmiloff-Smith, 1992). Differences and similarities between kinds of notations are discussed in seve-ral chapters (3, 8, 9, and 14). Finally, the extent and ways in which established scientists rely on conventional expressions and craft new formulatios in their publications are examined (Chapter 15, by Montemayor-Borsinger), throwing new light on the internal–external distinction and suggesting it might be better grasped in terms of a dynamic, constructive perspective.

EXTERNAL REPRESENTATIONS IN LEARNING AND TEACHING

In sum, all the chapters collected in this volume are in one way or another concerned with the dynamic relationships between internal and external representations in the context of learning and teaching. This seems to be quite a novelty. While there are examples of educational programmes that underscore the use of external representations to gain conceptual understanding (a paradigmatic example is Zoltan Paul Dienes’ mathematics education (Dienes, 1973), which relies on a varied repertoire of external embodiments to construct richer numerical structures), many educational programmes evidence a rather unidirectional approach towards the relations between internal and external representations. Also, educational studies have mainly focused on how internal representations of different kinds (Chi, Slotta & de Leeuw, 1994; diSessa, 1993, 1994; Limón & Mason, 2002; Pozo, 2003; Schnotz, Vosniadou, & Carretero, 1999) mediate the learning of cultural knowledge, which is inevitably expressed via external representations. But, as reviewed in this chapter and amply discussed in Chapter 17, the analysis of the simultaneously restricting and amplifying cognitive effects of external representations reveals that the relations between internal and external representations run in both directions.

In this last section we focus on the role external representations play in the explicitation and transformation of knowledge. We propose that the model of knowledge explicitation elaborated by Dienes and Perner (1999) — which, as the authors point out, is partly compatible with Karmiloff-Smith’s model of representational redescription (1992) — offers a good starting point to envisage how external representations take part in learning and teaching. However, two cautionary remarks are called for. In their original paper, Dienes and Perner did not concentrate on some of the concerns that are particularly relevant in this volume, such as cultural tools, academic contents, and teaching processes. Second, according to our view, their analysis is mainly aimed at identifying and connecting explicit and implicit processes in different cognitive domains, rather than at reflecting on the function of external...
representations in knowledge explicitation. Keeping these cautions in mind, we suggest that this model provides a useful tool to reflect upon the ways internal and external representations meet in the learning process leading to its enhancement.

Dienes and Perner claim that knowledge is a propositional attitude that may be decomposed into three main components: content, epistemic attitude, and self. Each of these components can be either explicit or implicit, albeit explicitation proceeds in a fixed order. The first component that can be made explicit is content, understood as “what the attitude is about” (Dienes & Perner, 1999, p. 737). To make content explicit is to predicate a property of an individual or thing. The second component in the path towards full explicitation is epistemic attitude. Attitude of knowledge is defined by “possession, accuracy, judgement and causal origin (justification)” (p. 739). The third and last component that can potentially be made explicit is self “as holder of an attitude” (p. 739). Thus, the distinction between implicit and explicit knowledge processes is posed to entail a gradient, with different levels according to which a given piece of knowledge may be either implicit or explicit. The authors provide evidence from different fields to sustain that progressive explicitation follows a fixed course, in such a way that content explicitation necessarily precedes explicitation of attitude, and explicitation of self requires attitude to have previously been made explicit (content → attitude → self). In our opinion, the core idea of progressive explicitation of knowledge according to a hierarchical order is relevant for the analysis of the roles of external representations in learning in educational settings. In what follows we illustrate different alternatives in which this may take place, by looking at findings of the studies collected in this volume.

To begin with, it is quite obvious that external representations allow learners to make the content of knowledge explicit, by parsing vaguely defined information and relating it to some kind of public and stable denotation — under the shape of gestures, spoken words, drawings, or some sort of notational nomenclature. The preschool curriculum presented in Chapter 5 shows that the opportunity to talk about shared experiences of observation or manipulation of everyday physical phenomena allowed very young children to formulate predications about individual cases and to identify properties of objects and situations. The motion sensor reported in Chapter 4 allowed preschoolers to become aware of the direction of their movements and thus convert them in an object to speak about, to draw, or represent by other graphic means. News in mass media directed the attention of children and adults to a partial solar eclipse occurring during long summer holidays, thus leading them to name, talk, and reflect about an event that otherwise would most probably have remained quite invisible; in turn, participants attained further levels of understanding when they went beyond the position of viewers or audience, and produced external bi- or tridimensional representations of the movement of the astronomical objects involved in this unusual phenomenon (Chapter 6). When the toddler observed in the longitudinal study in Chapter 2 showed two toys (one in each hand) at the time she said one, one, her awareness of perceptually different exemplars of a common lexical category as unitary and repeated wholes seemed to increase. Under the request to record freely how many necklace coloured beads each of three sisters had (Chapter 8), almost one-third of the youngest school
children produced a linear text reporting the number of beads of each colour belonging to each sister (without organizing the data according to any further variable). Via this format (which as the author points out is very close to oral descriptions) these second-graders stated properties of particular objects or collections thereof. Many other cases of content explicitation by means of external representations can be identified in the chapters presenting studies with young children, as it happens with the preschoolers who resorted to drawing in order to ‘put down’ the items to be shopped several days later — what in fact proved to be an adequate notational choice, since their figurative drawings supported subsequent recognition better than their current nonconventional writing did (Chapter 3). But this basic contribution of external representations to knowledge explicitation is also found in studies conducted with older children, adolescents, and even adults. For instance, in their written notes about plant growth experiments (Chapter 10), some secondary students recorded only ontological properties of the observed exemplars. In these cases, writing was limited to the explicitation of factual, general content (i.e., the kinds of entities being involved). University students who translated an algebraic formulation into a Cartesian graph by only placing one mark on the abscissa axis seemed to define a function in terms of a single property (Chapter 12).

In the instances mentioned above, external representations served the explicitation of content by bringing about awareness of facts, categories, regularities, or functions, thus enabling the learner to identify objects of knowledge and to represent their properties. At this level of explicitation, no doubts are cast upon what is known; knowledge is taken for granted (Hofer & Pintrich, 2002). We conjecture that this sort of naïve realism, which entails confidence in the truth and certainty of what is to be known or learnt, may be an essential ingredient for (developmentally or instructionally) very early knowing steps to arise.

Thus, on the one hand external representations can be conceived as primarily assisting the explicitation of content by equipping the learner with a repertoire of lexicons (gestural, oral, written, graphic, algebraic, etc.) that enable denotation and predication of objects and situations. On the other hand, the explicitation of epistemic attitude appears to enhance the adoption of a representational stance on knowledge. It ‘illuminates’ the knowing relation, making it possible to represent not only aspects pertaining to the object, but also the ways whereby the knowing subject or learner has an access to such an object and relates to it. From an educational point of view — and departing perhaps from a strict interpretation of Dienes and Perner’s proposal — we wish to include in this level the explicitation of relations among pieces of knowledge as well as relations among different representations of a common object, situation, or process. The contributions in this volume suggest that the use of alternative external representations to describe a single situation assists the explicitation of epistemic attitude, across developmental periods, learning situations, and domains of knowledge.

For instance, when young children in the participative school-based research presented in Chapter 7 used their bodies, produced drawings, and explained their graphic depictions in order to represent iron-, plasticine-, and polystyrenemade objects, they were not only rendering properties of particular objects explicit; they
were also coming to understand and compare aspects of the structure of matter. Similarly, in Chapter 4 we find that technological devices (a motion sensor and a graphic calculator), the ‘Prezzemolina’ tale, and their own gestures, drawings, and writings enhanced very young children’s awareness of relations among properties of motion and time. In turn, this augmented awareness enhanced what the authors call children’s ‘graph sense’ and allowed them to distinguish graphic representations of movement. As shown in Chapter 3, the drawings, writing, and numerical notations preschoolers produced as they prepared a birthday card and a shopping list reveal that they distinguished certain properties and affordances of these different external representation systems. Similarly, the examples highlighted in Chapter 6 show how four- to ten-year olds acquire and refine their knowledge about an eclipse by relating to different external representations. Along the same lines, the university students in Chapter 12 were found to derive a greater amount of conceptual inferences when they handled a wider range of external representations (textual, algebraic, Cartesian). In sum, using different kinds of external representations to account for a particular aspect of reality may be a way of highlighting that any particular external representation is partial and biased, and often complementary of other kinds of external representations, since it emphasises certain features and relations above others.

Following Dienes and Perner’s definition of the explicitation of epistemic attitude more strictly, we find that some contributions in this volume provide a picture of how educational or research practices may promote the learner’s (or participant’s) awareness of the kind of relation he/she is establishing with the object of knowledge at a given moment. This may have occurred when preschool children were encouraged to justify their statements in the frame of preschool science activities (Chapter 5), when preschool and primary school children were invited to compare the relative ease they found in drawing or writing to achieve a particular communicative purpose (Chapter 9), or when secondary school students discussed the meanings of the word ‘infinite’ (Chapter 13).

Thus, the use and comparison of alternative external representations, as well as relatively situated metacognitive talk (Pramling, 1996) may aid the explicitation of knowing or epistemic attitude. As discussed in Chapter 13, the presence of different semiotic languages and representations can facilitate the contact with different perspectives and hence externalize epistemic attitude as a knowing stance, rather than as a given, fixed, and certain entrance. Nevertheless, the difficulties involved in learning external representations and getting to use them flexibly as mediational tools, just as well-documented translation errors (see Chapter 12 for mathematical representations and Chapter 14 for chemical codes), indicate that the potential of external representations for knowledge explicitation is related to a wide range of factors, including learners’ previous domain knowledge and the functions external representations are linked to in context of the particular activity. Some chemical codes are learnt and used more easily than others, some kinds of graphic representations tend to be better understood than others, and some means of numerical notation are more useful than others for particular ends (for instance, the use or interpretation of tallies in one-to-one correspondence require no or very little instruction,
while difficulties in grasping the base 10 number system persist into mid-elementary education, Scheuer et al., 2000). These scientific and cultural representational systems that do not easily spread out socially seem to be particularly incompatible with human implicit cognition (Sperber, 1996) or embodied knowledge (Pozo, 2003), or entail enduring epistemological obstacles (Bachelard, 1938). However, these highly compacted systems, with principles that may be difficult to grasp and internalize, often have a very broad representational potential.

Different kinds of external representations (writing, numerical notation, maps, tables, graphics, charts) represent particular aspects, angles, or layers of reality (linguistic contents, quantity, spatial localization, relationships between variables) according to a set of specific restrictions that are conventionally regulated. Some external representation systems are difficult to learn in a fully spontaneous way, due to their high level of formalization and conventional nature (Garcia-Mila & Andersen, 2007; Sfard, 2002). These difficulties are especially patent in the learning of science and mathematics. In these areas, instruction often emphasizes fragmented components and the formulation of preestablished rules (focusing on the explicitation of content, following the terms used by Dienes and Perner), and it may be quite hard for learners to make relational aspects explicit, such as the links between components, the structure of the system, or the fluid relations between declarative, procedural, and conditional knowledge. Difficulties of this sort are examined in Chapter 11 in relation to geometry. Some mistakes made by both children and adults when requested to apply labels of geometrical shapes may stem from the interference with everyday meanings, which most probably are of a more implicit nature, and thus more pregnant than formal labels. It may also be the case that some of the mistakes made by children in these tasks were motivated by the rejection of nonprototypical items. Prototypicality is based on similarity, which in turn supports a common mechanism of implicit learning (see, for instance, Pozo, 2003; Reber, 1993; Rosch, 1978).

According to Dienes and Perner, the third level in the process of knowledge explicitation is gaining awareness of the meaning of knowledge in relation to self, and of changes and continuities in own knowledge and views. We appreciate that, in this leading paper, this ultimate level of explicitation has received less attention than the other two. According to the authors’ rather brief explanation, explicitation of self is related to full consciousness and is hence the last step on the way from implicit representations to fully explicit knowledge. “The self is an experiencing agent in possession of the knowledge” (Dienes & Perner, 1999, p. 752).

How can external representations contribute to externalizing the knower’s or learner’s self? A very nice example can be found in the interview with a scientist presented in Chapter 15 in the context of producing and communicating new understandings, rather than that of learning already validated knowledge (as the other contributions in this volume deal with). The scientist highlighted that a major difficulty he or she experienced in writing a scientific paper was getting to flesh out the relation between his or her view about physics and physics in general. In our opinion, explicitation of self entails placing one’s own perspective in relation to the perspectives expressed by other persons, communities, or identifiable trends of
thought. This level of explicitation allows people to think about their knowledge as a personal attempt to grasp their natural, social, and/or subjective environments (an example of this can be found in Chapter 9, when preadolescents expressed that drawing and writing about themselves required communicating a constructed representation, or perspective, to others or to oneself). Thus, explicitation of self as ‘holder of knowledge’ opens the possibility of considering such knowledge as being relative, transient, construed, and, hence, of connecting it to broader frames of reference. As the film director Kieslowski expressed in one of his documentaries — he too was taking up Virginia Woolf’s appreciation of description, with which we opened this chapter — ‘living in an undescribed world is hard. (...) It’s like having no identity. (...) To put it more radically, you feel completely cut off from other people. (...) You are alone.’ (Kieslowski, 1995, as cited in Cummings, 2006).

Awareness of the knowing self may be facilitated by establishing contrasts or discrepancies (Bartsch & Wellman, 1994) and drawing connections within an ontogenetic frame of reference and/or within an interpersonal frame of reference. When self gains some degree of explicitation in an ontogenetic frame of reference, the learner becomes aware of the changes and continuities in her ways of learning, thinking, understanding, and communicating about knowledge of something over time. Such present view may thus be, at least under certain circumstances, experienced or conceived of by its holder as a lens that is distinct from and related to past ways of knowing, and at the same time somehow prefigures forthcoming, emerging views. When the explicitation of self is extended within an interpersonal frame of reference, distinctions and connections are recognized among the personal views and those of others, they are individual (as we see in the discussions about the concept of infinite in Chapter 13) or more or less validated collective thought trends (as the scientist interviewed in Chapter 15 formulated).

In sum, external representations are essential to construct knowledge, refine it, modify it, share and appropriate it. Though contemporary educational environments are deeply concerned with the transmission of external representation systems, the enormous social and cognitive potential of these symbolic tools seems to remain rather unexploited and underestimated. First, external representations present learners the opportunity to express or externalize aspects of internal representations. Second, content externalization may be a requisite to reflect upon attitude or knowing processes. Moreover, different kinds of external representations pose restrictions that direct the learners’ attention to some traits of the objects externalized, at the time that other properties remain partly hidden or obscure. Becoming aware of the affordances of different external formats may help to attain awareness of different relations and perspectives that, subsequently, may contribute to develop fuller consciousness. Fully conscious knowledge entails an awareness of the ways in which knowing agents relate to knowledge, as well as of the impact knowledge has on the development of the person who learns. In the process of knowledge construction, the explicitation of content, attitude and self may operate in a sort of spiral. Emergently explicit aspects of a higher level as attitude (knowing processes) may subsequently come to form part of given content knowledge, via processes of
EXTERNAL REPRESENTATIONS AS LEARNING TOOLS

systematization and nominalization as those fleshed out in Chapter 15 in reviewing Halliday’s Type 1 referring or ‘distilling’ grammatical metaphor.

The core of this volume is the result of two symposia presented in the 11th European Conference for Research on Learning and Instruction (Nicosia, 2005). Other scholars with common interests were invited to join into this editorial project, thus opening reflection and discussion further. External researchers working in the field (fully listed in the volume’s Acknowledgements) were then invited to review the different contributions. Their generous and thoughtful engagement certainly contributed in assisting the work of the authors. The willingness of Katherine Nelson and Ricardo Nemirovsky to reflect on the contents of this volume and to develop their reflections in writing will definitely assist readers to reconstruct their own appreciations, discoveries, and uncertainties. We warmly remember Gyoo Hatano, whose participation would have positively enhanced this project. We deeply regret his passing.

NOTES

1 Authors listed alphabetically

REFERENCES


EXTERNAL REPRESENTATIONS AS LEARNING TOOLS


Sfard, A. (2002). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. Educational Studies in Mathematics, 46, 3–57.


ACKNOWLEDGEMENTS

We thank Zoltan Dienes and the participants in this book (authors of chapters and editors) for their reflective comments on a previous draft. María del Puy Pérez Echeverría’s work was supported by Ministerio de Educación y Ciencia de España (SEJ2006-15639 C02-O1) and Nora Scheuer’s by Universidad del Comahue (B-139), ANPCYT (06-1607), and CONICET (PIP 5663).

María del Puy Pérez Echeverría
Facultad de Psicología, Universidad Autónoma de Madrid

Nora Scheuer
Centro Regional Universitario Bariloche, Universidad Nacional del Comahue and CONICET
NORA SCHEUER AND ANNE SINCLAIR

2. FROM ONE TO TWO

Observing One Child’s Early Mathematical Steps

INTRODUCTION
Developmental psychologists have conceptualised mathematical entities, number concepts, numerical competence, mathematical thinking or knowledge, and the associated activities such as counting, calculating, measuring, and quantifying relationships in very diverse ways. The confused variety of implicit definitions, epistemological viewpoints, and research methods evident in the field of the young child’s mathematical development or learning can be seen as due to several related issues.

No consensus concerning the nature of mathematics exists, be it within mathematics itself (see Hersh, 1997) or in philosophy and epistemology. The fundamental nature or properties of natural number have been described in different ways by mathematicians, classical and modern (see Droz, 1991).

The contrast with another area of child learning, reading (usually also deemed to be symbolic), helps us perceive the mysterious and self-referent nature of mathematics. Indeed, in the area of reading, the structure of spoken language and its symbolisation or materialisation in writing systems, and the cognitive and cultural effects of literacy, have been much discussed and are quite well understood (Olson, 1977, 1994). The cognitive and perceptive capacities children deploy when learning to read and write can be delimited, and their progress is relatively transparent and comprehensible to the psychologist’s eye (Fitzgerald & Shanahan, 2000).

Due to the ill-defined nature of the content children grasp when they come to acquire numbers and manipulate them, developmental psychology has adopted approaches that are formalist, fundamentalist, and/or highlight the role of symbolic construction in mathematical knowledge. Formalists have used criteria elaborated within logic or mathematics to define what the roots or building blocks of mathematical thinking might be. Piaget & Szeminska (1941), for example, inform us that the ‘number concept’ is forged by the welding of classification and seriation schemes, and that the litmus test of grasp of the number concept is being convinced of numerical invariance in instances of the physical displacement of small objects that make up a material collection (number conservation).

To be sure, a full-fledged number concept, must, by definition, somehow include the understanding that subsequent numbers in the natural number series represent ever larger quantities, and that a collection of ‘five apples’ represents a set whose extension is precisely 5, that contains elements that can be ordered, etc. However,
Piaget’s account still awaits confirmation, as when and how the operations required for seriating physical objects and passing classification tests meld to create the grasp of number has not been brought to light. The number conservation task itself seems to call on physical knowledge and mathematical thinking simultaneously. It concerns particular properties of some of the objects in our physical world, combined with our conventional ways of viewing and publicly describing them.

Similarly, counting has been unpacked in a formal way to track its essential components (Gelman & Gallistel, 1978). Certainly, one-to-one correspondence is a necessary procedural and conceptual part of the act of counting. Yet, one-to-one correspondence in spontaneous action or play, such as putting one stick in each cup, precedes correct counting by years (Sinclair, Stambak, Lézine, Rayna, & Verba, 1989). But, how does one-to-one correspondence come to be used to mathematical ends, to be implemented with an ‘abstract’ content rather than a functional or figural one? How does the scheme contribute to an understanding of cardinal number (Fuson, 1998; Sophian, 1997)? To tackle puzzling problems of this nature, more closely related to observable behaviours where numbers or any type of quantitative or mathematical elements are manipulated by subjects, many researchers have used an approach of a more functionalist type, and/or studied children’s spontaneous or semidirected behaviour in more open tasks (for example, Saxe, Guberman, & Gearhart, 1987; Sophian, 1996).

Let us turn to the fundamentalists. In the past 25 years a research trend has attempted to show that human infants possess ‘innate’, inherent, or wired-in ‘core’ numerical knowledge (Carey, 2001) or ‘number sense’ (Dehaene, 1997). Studies have shown that infants automatically respond to a variety of presented materials (static or moving objects or representations thereof, actions, sound stimuli) in function of their numerosity. They react to differences and thus appear to be discriminating between presentations of, for example, 2 vs. 3 and sometimes 3 vs. 4 stimuli (see, for example, Antell & Keating, 1983; Bijeljac-Babic, Bertoncini, & Mehler, 1991; Starkey & Cooper, 1980; Strauss & Curtis, 1981; Van Loosbroek & Smitsman, 1990; Wynn, 1996; or, for larger quantities, Xu & Spelke, 2000). In most of these studies, the human number sense is not related to learning mathematics or, indeed, to any other capacities or skills. On occasion one feels that learning is presumed to be a matter of acquiring arbitrary rules (mathematics à la Wittgenstein) or absorbing what schools offer. Naturally, addressing or studying every aspect at once is not possible; and yet, the infant’s ‘number sense’ remains somewhat disembodied, acontextual — how cognition and culture may meet is not explored.

Lastly, some authors, seeking to integrate the biological and cultural in mathematical cognition through human symbol use, have proposed that it is our capacity to generate, appropriate, and communicate with symbols fluidly that allows us to go beyond biological dispositions, both for extension (grasping magnitudes), and systemic integration — e.g., relating one number to another in various ways instead of simply distinguishing collections of XX from collections of XXX (see Dehaene, 1997). Within our developmental perspective, the grasping of numbers, or the learning of mathematics, involves becoming part of a numerate culture.
(Bishop, 1991; Nunes & Bryant, 1996; Sophian, 1996) and learning to use and manipulate particular symbolic concepts and procedures (natural number, counting sequence, the idea of subtraction, of infinite strings…) that have been constructed by human cultures gradually over a long time span, from the Upper Paleolithic onwards (Ifrah, 1994).

These symbolic, or perhaps more properly, purely conceptual structures (the number 18 has no referent and is but an element in an abstract system) are necessarily culturally transmitted. According to Hurford (1987), number concepts could not exist without language. We may take him to mean not only that cultures or groups that don’t possess number words (beyond, for example, *three*) don’t indulge in mathematical activity such as measurement, but to alert us to the fact that without the construction of an abstract frame of reference, or a structural schema that is embodied, translated, or brought into being in a material way (words, standfors, artifacts such as counters, etc.) that may serve as tools for learning, thinking, acting, and communicating (required for further co-construction), numerical thinking and activity will not develop, in individuals or groups.

It is a truism to state that these constructions are related to, built on, or rest upon, both bedrock human perceptive-cognitive capacities and the nature of human action, as well as characteristics of the physical world, as humans live in that world. Our brains may be seen as having evolved from patterns of real world dynamics and structures (Vandervert, 1994); partial isomorphism between reality and mathematics may exist (Piaget, 1967), mathematical thinking may be an abstraction of action and action-schemes (Lakoff & Núñez, 2000), or more generally of human and animal perception as it functions in activities such as hunting prey (Wynn & Bloom, 1992). In all such speculative discussions, some distilling of experience, some autoreferential cognitive mining, is inferred to occur. For the result to be mathematical, it must ultimately find expression in some structural scaffold.

Our study was prompted by curiosity. We wondered what children’s first, very early, production or use (comprehension, uptake, etc.) of numerical or mathematical symbols or representations was like, and if it was possible to follow progress and characterise learning in some way. We wanted to understand the child’s spontaneous interest and efforts, rather than test its performance at some point in time. We thus carried out an observational and longitudinal study: As parents, we observed two of our own children, a boy (L) and a girl (C), from the age of 18 months, using the only method available, paper and pencil recording. In this chapter, we report and discuss only the very early observations of one child, the girl C.

Describing nature, in this case a special type of ongoing symbolic activities embedded in a cultural niche, has gone out of fashion in mainstream developmental cognitive psychology. Yet, studies performed in the early 1900’s convinced us that the material gathered would be rich and lend itself to interpretation. In particular, Decroly and Degand (1912), working on continuous and discontinuous quantities, followed a little girl from the age of 13 to 54 months, using observations and naturally inserted test questions or actions. Her progress can be tracked, even if it appears patchy and some novelties seem to appear *ex nihilo.* (Fischer 1991, p. 238, gives a partial list of older studies). We hoped that data of this kind might spur
reflections concerning the connections between the different approaches briefly reviewed above.

THE STUDY: FOCUS, PROCEDURE, CONTEXT, AND PARTICIPANTS

Our observations focused on the very first episodes involving any kind of recognisable external representation of number in which the child’s participation went beyond listening or observing. For C, this occurred at 20 months of age. We attempted to gather all such episodes, independently of what type of event initially brought the numerical external representation into the scene — the child, another person, material artefacts, or events that were merely observed by participants.

All episodes were written down as amply as possible on the spot. On some occasions, inevitably, only notes could be taken; these were completed later the same day. Since we thought it important to preserve the natural quality of family interaction, as well as to avoid ‘preforming’ the observations as much as possible, we used a distancing procedure. No interpretative framework was set up beforehand, no analyses were carried out, and no discussion took place during the observation period. We acted as scribes and took up this material only when the children were well on their way in primary school.

The girl (C) was observed from the age of 1 year and 8 months (1;8) to 2 years and 10 months (2;10). C lived with her parents (O will be used for mother, who performed the observations, and F for father) and her brother (B), who was 3 years and 3 months older. The family spoke Spanish. C attended a day care centre 5 days a week — no observations were recorded during those times. Until the age of about 14 months, C’s communication relied heavily on facial and bodily expressions; she began to speak comprehensibly at around 16 months of age. During the preschool years following the observation period, C showed no special interest in numbers, mathematics, or related topics. She completed primary school with ease, with good marks in math.

In this chapter we report the first 14 observations, stretching over 3 months, in extenso, using plain, descriptive terms, so that the readers may have the opportunity to interpret for themselves, even if the data have inevitably been filtered by the observer’s lens. The original formulation in Spanish is provided when it conveys a particular shade of meaning that is not readily translatable into English. The comments that follow some of the observations were written years later and serve to guide the reader, hint at, or explain our interpretations.

Table 1 presents a classification of the observations based on our present focus, the child’s representational stance (see A. Sinclair, 2005, for a sociocognitive analysis of the observations of the boy L). It gives the number and kind of entities in the referenced array or collection, the type of observable instantiation (verbal, gestural, both), the numerical function the observable or external representation is oriented to, as well as the first time it was observed.
Table 1. C’s external number representations: observations that document them, number and kind of entities in the referenced array, type of observable instantiation, main numerical function, first occurrence in terms of C’s age

<table>
<thead>
<tr>
<th>External representation of number</th>
<th>Observations</th>
<th>Referenced array</th>
<th>Type of observable instantiation</th>
<th>Main numerical function</th>
<th>First time observed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Number of entities</td>
<td>Kind of entities</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>Chunks of time</td>
<td>Verbal</td>
<td>Measurement</td>
</tr>
<tr>
<td>One, one, one scheme</td>
<td>2, 3, 10, 14</td>
<td>3</td>
<td>Present objects, or just seen</td>
<td>Verbal-cum-gestural, verbal</td>
<td>Enumeration</td>
</tr>
<tr>
<td>One, one’ scheme</td>
<td>3, 6, 7, 11</td>
<td>2</td>
<td>Present objects, embodied signifieds (Obs. 11)</td>
<td>Verbal-cum-gestural, verbal</td>
<td>Enumeration</td>
</tr>
<tr>
<td>Global appreciation</td>
<td>4</td>
<td></td>
<td>Approximate large quantity</td>
<td>Present objects</td>
<td>Enumeration</td>
</tr>
<tr>
<td>One</td>
<td>6, 8</td>
<td>1</td>
<td>Present object</td>
<td>Verbal-cum-gestural, verbal</td>
<td>Quantification</td>
</tr>
<tr>
<td>Response to cardinal request</td>
<td>8</td>
<td>2</td>
<td>Available but not visible objects</td>
<td>None</td>
<td>Quantification (cardinal count)</td>
</tr>
<tr>
<td>Two</td>
<td>8</td>
<td>Unclear reference</td>
<td>Unclear reference</td>
<td>Present actions</td>
<td>Verbal</td>
</tr>
<tr>
<td>Three / finger pattern</td>
<td>9</td>
<td>3</td>
<td>Past actions</td>
<td>Verbal-cum-gestural</td>
<td>Quantification</td>
</tr>
<tr>
<td>Two / finger pattern (thumbs)</td>
<td>11, 12, 13</td>
<td>2</td>
<td>Embodied signifieds, imagined objects, combination of a present and a recalled object</td>
<td>Verbal-cum-gestural</td>
<td>Quantification</td>
</tr>
</tbody>
</table>

THE OBSERVATIONS

Getting Started

C’s first production of number words (as recognised by O) is promoted and structured by her brother in the context of social physical play, when both children are running races on a slope. Numerical expressions are directed at measuring time, to mark when to start running.
B to C: I'll teach you to run. You must lift up your legs, but more. Look, like this. B runs, C follows him as best as she can. B: Again, let's see. One, two, three. C: Three! (¡Tés!, for tres.) Both of them run. They repeat the whole sequence several times, with B saying before getting started: One, two, three, and C, at the same time: One, three! (¡Uno, tés!). Although C says only two number words (first and last), she starts running when B enunciates the third word. B complains to O: She can't wait for the two! The game goes on. B: Now, something very difficult. Ten! C repeats: Ten! (¡Die!, for diez.) B: Nine! C: Nine! (¡'Eve!, for nueve.) B: Eight! C: Eight! (¡Oto!, for ocho) and, though B stands still, she starts running. B: No, not yet! (He is apparently planning to count back down to one or zero.) The game is dropped.

The first observation fits a Vygotskian description of learning in informal settings very neatly. An older, more competent child introduces the younger one to a new, structured, interactive activity, using symbolic forms. B organizes the transmission so that C's cognitive effort is reduced, as evidenced when he highlights novelties (Now, something very difficult) and separates the new information into chunks (one number word per conversational turn). C doesn't just imitate, repeat, or follow orders. She takes into account an underlying, abstract structure that is not explicitly highlighted as such by B. When her brother counts backwards, C starts running at the third number word, eight, which suggests that when she repeats these different number words, she is invisibly keeping track of a three-item sequence. Just as one word is made to correspond to one step, three words are made to match to a three-step string, independently of what the specific words are. Though both children seemed to enjoy their game, they do not take it up again in the following weeks. C is not seen participating in any activity involving number symbols until 1 month later.

Two and Three Units as Cohesive, Bounded, and Repeated Wholes

The next episodes observed concern a foundational theme in the domain of number: units. In sharp contrast to Obs. 1, in the second episode observed C uses number symbols to give shape to what seems to be an inner driven, reflective activity. She suddenly becomes interested in a new working space.
Obs. 2.  1;09;12.

C has just begun to use a pot. Sitting in the toilet in this position, she seems to be captivated by a new sight. She stares at the floor (a 3 by 2 metre surface covered with hexagonal, red tiles) and points (with her index) successively at three tiles in front of her, saying: One, one, one. Correspondence between word, conventional gestural point to tile, and thus tile, is smooth and correct. During the next 6 weeks, C frequently points to any three tiles close to her, in any order, and says one, one, one, establishing a precise oral-gestural-object correspondence, as reported in Obs. 2. Her attitude is descriptive, ‘matter of fact’. We may wonder why C repeats this activity in the same context, or why the regular, fixed display continues to provoke her interest. Perhaps, she feels that there is more in the display than she can capture, so that the view of the tiled floor provides an open task to which she returns again and again.

The day after Obs. 2, C adjusts the one, one, one scheme to two objects.

Obs. 3.  1;09;13.

C comes to O’s bedroom, holding a toy elephant in each hand. Both elephants are grey; one is made of wood and tiny, the other one is larger and made of plastic.

C: Look, Mum! Phant! (Fante, for elefante.) One (looks at one elephant), one! (Looks at the other one.)

O: How nice! What if you bring two giraffes now?

C looks at O, seems not to understand and turns to play with the elephants.

The adult’s request for two giraffes is not intended as a test of any kind. Rather, it indicates that O has automatically interpreted one, one as ‘two’ and behaves as if this equivalence is natural for C as well. As shown in Table 2, C introduces several changes in her first reuse (as it arises in our observations; she spent the day at the day care centre) of the enumerative scheme she has put into words and action the day before. Other differences concern the perceptual access to the enumerated entities: the elephants can be, and are, manipulated. Moreover, whereas the tiles are perceptually identical — considering them to have perfectly similar attributes does not seem to require any parsing or abstraction; we suppose that it was their very ‘sameness’ and the part-part-whole relationship they entertain with the plane of the floor that prompted her observation. By contrast, the elephants are made identical (comparable, similar, etc.) through object classification and/or lexical knowledge.

C uses the one, one scheme intensively during almost a month; it is observed once, twice, even three times a day. She frequently picks up two toys, one in each hand (simultaneously or successively) and states: One, one!, demonstrating to a family member, looking very pleased. In all cases, the objects are small, fit easily into her hand and are either perceptually identical (e.g., two cups of the same set),
Table 2. Differences between the entities being enumerated in Observations 2 and 3

<table>
<thead>
<tr>
<th>Observation 2</th>
<th>Observation 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three</td>
<td>Two</td>
</tr>
<tr>
<td>Red, flat geometrical shapes</td>
<td>Grey, three-dimensional objects</td>
</tr>
<tr>
<td>Items are not labelled with a category word</td>
<td>Items are labelled with a category word (phant)</td>
</tr>
<tr>
<td>Glued on a continuous surface, do not afford being handled</td>
<td>Small individuals held in one hand each</td>
</tr>
<tr>
<td>Perceptually identical</td>
<td>Individuals differ in their shape, size and material</td>
</tr>
</tbody>
</table>

or are dissimilar but belong to the same object category (e.g., two hair bands). Usually, C picks the toys from a larger assortment (30 or 40) in a big box in her bedroom. The fact that C applies this scheme almost exclusively to her own belongings, with evident pleasure, suggests she may be stating something related to entitlement, of the kind: Look, I have two x! (Carr, Peters, & Young-Loveridge, 1994). At the same time, she is exploring classification and enumeration in ways we can only guess at.

Only once during this period is C observed directing her attention to the joint presence of very many small and similar objects. She uses a global quantifier.

Obs. 4. 1;09;25.

In the park, C looks at the ground, which is covered with pebbles: How much stone! (¡Cuánta piedra!)

Almost 3 weeks after the first observed actualisation of the one, one, one scheme, C is observed applying it to three items in contexts other than the bathroom tile floor.

Obs. 5. 1;10;04.

C, in her stroller, B and O, walking in the street, pass in front of a toy shop, without stopping.
C: Bucket! Bucket!
O and B look for the buckets in the shop window without success; the three hanging buckets are only visible from C’s low position.
O: Where? I don’t see any bucket.
C: Bucket!
O kneels down, and notices three buckets hanging from the roof.
O: Ah! She continues walking.
A few seconds later, when the buckets are no longer visible, C says: One, one, one.
B (looking perplexed): What?
O (to B): *Instead of saying 'one, two, three', she says 'one, one, one'.*

B: *Ah, as men did very long ago.*

The topic is not elaborated further.

Once more, C’s attitude is descriptive, constative, assertive. The buckets, as physical objects, are not pertinent to ongoing action in any way; but the reading of reality merits expression. B’s perplexity signals that C is formulating numerical insights in her own way, as opposed to adopting conventional forms.

**Obs. 6. 1;10;05.**

O and C are playing, as they rest on C’s bed.

O: *How many hands have you got?*

C (as she lifts and rotates her right hand): *One!* In doing so, she looks at O’s face — not at her own hand. C continues rotating her right hand, when she lifts and rotates her left hand as well and, still looking at O, says: *One!* She smiles, looking very pleased.

We interpret another shade of meaning is added here. Hands have an existence of their own; each hand is independent of the other; one entity or exemplar exists without belonging to a kind or class in the way objects like cups and buckets do. In our eyes, C is not happy that she possesses two hands, but is pleased with her clever answer and the interpretation it rests on — hands can also be described as *one, one.*

**Obs. 7. 1;10;05.**

C asks to go to the toilet, but her panty is wet already. While she is sitting on the pot, O fetches a clean panty. The wet panty is still on the floor.

C: *One, one.*

O (at first is perplexed, she does not understand what C is referring to): *One, one?*

C: *Yes!*

O: *Ah, the pants!*

C nods emphatically.

These observations bring to light the emergence, consolidation, and expansion of a representational scheme dealing with units and one-to-one correspondence. Schemes have been defined as ‘the invariant organisation of behaviour (action) for a certain class of situations, made up of procedural ingredients as well as representational ones’ (Vergnaud, 1996, p. 222, our italics). From this standpoint, C’s *one, one* and *one, one* are schemes.

How to describe or define the kinds of displays C picks out to apply these enumerative schemes? What is it that three visually identical, flat, red tiles glued on a larger continuous surface share with two different toy elephants? What is the similarity between pants, one dry, one wet, and rotating the left hand and the right
hand? What the descriptions of these objects have in common is an abstract, formal structure C relies on to describe the presence of repeated, co-occurring units, or coherent wholes that can be counted as one, to paraphrase Baruk (1992, p. 758 and ff.). In agreement with many experimental results regarding infants’ ‘numerical’ competencies prior to speech, as well as studies with older children and adults on easy numerical quantification of discrete entities, evidenced in action or ‘at a glance’ and sometimes coupled with verbal descriptions (see Fischer, 1991), these observations hint at a limit for C’s use of her enumerative schemes: collections under four.

Schmandt-Besserat (1992), as well as Hurford (2001), has provided evidence that both linguistic and anthropological data suggest that humans are intrinsically or ‘innately’ limited to perceptively differentiating collections of under three (X, XX, XXX are perceived as different), as well as distinguishing all these cases from collections of four or more.

By contrast, C represents her awareness of indefinite, perhaps endless, repetitions by other means; a global quantifier (see Obs. 4). We may wonder why she treats the pebbles in the park in a different way than she deals with a surface covered by tiles, or toys from a box. Certainly the displays or conditions of viewing play a role: a few pebbles on a plate on a table would surely not call up the same treatment. A few noncontiguous items would be treated as a collection of discrete objects, whereas a large ground covered in pebbles becomes, we assume, a ‘plurality’, or an ‘open-ended string of unitary items’ (see von Glasersfeld, 1981, pp. 86-87). Quantification in numeric terms is not sensible or appropriate — some other measure is needed. Tiles and toys, on the other hand, as collections, are mentalised as existing within some boundary (walls, box, or imagined numerosity, extension, of the collection or set) and thus are deserving of individuation and enumeration. Let us note that C uses the singular piedra; at this time she has not acquired linguistic plural forms, possibly because plurality itself (as a quantitative judgment, e.g., 1 vs. 2 vs. ‘many’) is not yet conceptualised by her.

FROM ONE, ONE TO TWO

In the span of a few days, a noticeable change in C’s ways of referring to two objects comes to light.

Obs. 8. 1;10;05. (Though this episode occurs in between observations 6 and 7, we present it here for the sake of clarity.)

O: Would you bring me two dishes? (Referring to C’s playing stuff).
C (Brings a casserole and a bowl — both are dishes for her): One, one.
O: Great, two dishes.

C and O pretend they are eating. Then C continues playing on her own. She lays an imaginary table on the floor. When she is no longer handling the two dishes, she sings: Two dishes, two dishes! (Do’ plato, do’ plato!)
(...)
O: How many pacifiers have you got? (Her pacifier is not near at hand).
C (laughs): One! (Correct.)

In this case O initiates the interaction with the deliberate purpose of exploring how C deals with a cardinal request involving the number word two. Here C deals with the number words two and one in different ways. First, C reacts naturally and correctly to O’s request (see Figure 1, arrow a), by extracting two dishes from a larger, disordered lot — dishware and other toys in her box. Next, C describes the collection with her usual scheme (arrow b). C ties both formulations (two in comprehension and one, one in production) to the same collection, but there is no evidence that she establishes any relation between the formulations. We cannot know if she processes two automatically without reflection or awareness and then proceeds to produce her own description as divorced from her previous processing, or if she is, by her actions, setting up some kind of correspondence (such as a more general meaning equivalence) between two and one, one. The difficulty is due to the fact that this is C’s first reaction to the word two, and she has never produced it.

Next, C takes up the string two dishes as a song as she happily goes on playing, without ostensive reference to objects (arrow c), which might be taken to indicate that C was aware of some novelty. The last two turns in Obs. 8 indicate that the contrast between one and one, one (and/or possibly two), that is between collections composed of two element and unitary objects (or ‘collections’ of 1), is well drawn.

![Figure 1. Relations presumably established by C in Obs. 8.](image)

A day later, an episode that will have powerful effects takes place.

Obs. 9. 1;10;06.

C, B and F have just come back from an amusement park where they have spent the afternoon. When C is telling O about a particular ride, F intervenes: We went three times, and he lifts his thumb, index and middle fingers simultaneously. C immediately repeats: Three times (Te’vece’) and attempts to reproduce F’s finger pattern. This proves very difficult. C has to use her left hand to keep middle and index fingers straight up and close to the thumb.
Yes, like this, three. When he notices C’s imitative behaviour, he deliberately maintains the finger pattern, showing it to C and smiling approvingly at her attempts.

C keeps looking at her finger pattern and showing it to her parents, looking very pleased with her accomplishment.

The next day, C continues to apply the ‘one, one, one’ scheme to three objects of the same kind.

Obs. 10. 1;10;07.

C is pretending to cook. She picks up a dish, another one and yet another. She is holding them tightly so that they don’t fall when she says: One, one, one.

Two days later, C is heard to produce the word two in relation to a collection of two.

Obs. 11. 1;10;09.

O, B, and C are getting ready to go to the public library for the first time after the summer holidays. (During her last visit to the library, 3 months earlier, C covertly took two books and made her way to the exit. The librarian saw her and promised her she could become a library member after the holidays.)

O to C: We’re going to the library. We’ll look for two books. Not only for B, for you too, C.

C immediately gets up and lifts her two thumbs keeping them straight together, saying at the same time: Do’ biblo, do’ biblo! (Two ‘libre’, two ‘libre’! Libro — not biblo — is the correct word for book in Spanish.)

Grandmother (looking at C’s lifted thumbs): Fine, two books! (¡Qué bien, dos libros!)

Standing still, C looks closely at her two thumbs and draws them slightly apart. She continues looking at them attentively as she says: One, one.

C uses the word two for the first time (as far as the observations show), taking it up from O’s previous utterance. At the same time, she illustrates, or fleshes out, or sets out to discover, determine, the meaning of do’ biblo/dos libros, or of the word two by adopting — and adapting — the representational medium or method (fingers) shown to her in Obs. 9. C transforms both the signifier and the signified (see Table 3): (Any) two books vs. three repeated past joint actions; a different type of finger pattern, while conserving the principle of one finger for one represented entity.

Whereas the ‘one, one, one’ and ‘one, one’ schemes appear to be inner rooted or individually constructed, this new behaviour seems to be a clear case of internalisation of an ‘external’ form that has appeared first at an interpersonal, social level (Vygotsky, 1978; Wertsch, 1985). It seems that her thumbs offer C ‘perceptual, proprioceptive (and) representational material’ (von Glasersfeld, 1981, p. 87). It is C’s own thinking or schemes that determine the salience of the characteristics extracted and represented (thumbs as co-occurring units), suggesting an instance of reflective abstraction, or pseudoempirical abstraction. In von Glasersfeld’s terms (1995):
FROM ONE TO TWO

… ‘the focusing of attention, not on sensory-motor signals, but on the results or products of prior attentional operations. Something that has been constructed by means of an attentional pattern is now reprocessed and used as raw material for a new sequence of focused and unfocused pulses. In the case of unitary items, this creates an abstract or arithmetic unit that, in our view, represents Piaget’s ‘element stripped of its qualities’ (p. 103).

The operation carried out results in a particular example of early word learning, as the leap from the unconventional if comprehensible one, one to two implies a degree or type of abstraction different from sorting out ‘birds’ vs. ‘planes’. Transliterating two biblo into one, one is an authentic redescription (Karmiloff-Smith, 1992) that has bidirectional power. It is the first one she was observed to make in this area.

Table 3. Differences between C’s first attempt to appropriate an oral-finger external form of representation and her first reuse of it

<table>
<thead>
<tr>
<th>Observation 8</th>
<th>Observation 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three</td>
<td>Two</td>
</tr>
<tr>
<td>Items are not labelled with a common category word</td>
<td>Items are labelled with a common category word (biblo)</td>
</tr>
<tr>
<td>Repeated past actions</td>
<td>Anticipated, unspecified objects</td>
</tr>
<tr>
<td>Amusement park</td>
<td>Library</td>
</tr>
</tbody>
</table>

C has slipped into a new symbolic level, by transliterating meanings or making redescriptions, by using a new kind of external representation (fingers), and by using an abstraction procedure that has turned a signifier into a signified (fingers as signifieds that can be contemplated) ‘bypassing’ direct reference to the physical world (Goldin & Kaput, 1996), all at once. Here, she has accomplished what remained sketchy or fledgling in Obs. 8, where the relation between one, one and two dishes was temporally contiguous and identical in reference to physical objects, but not more.

Thereafter, during the following weeks, C frequently refers to two perceptually different objects in the same category (e.g., bracelets, elephants, giraffes, Duplo men, books, etc.) by lifting two thumbs as in Obs. 11 (thumbs separated) and saying two. The ‘one, one’ scheme is abandoned. The objects need not be present. Some examples:

Obs. 12. 1;10;21.

C to O: Biscuit! (¡Lleta!)
O gives her a biscuit. C complains and doesn’t take it: Two llela!, lifting both thumbs.
Walking down the street, a bicycle passes by. C: *Bis, bis!* (For bicycle.) A few minutes later, another bicycle passes by. C: *Bis!* She looks at O and lifts two thumbs, saying: *Two bis.*

Once C is a library member, C’s parents limit book reading at bedtime to two books. From age 1;10 to about her second birthday, when O or F say: *Let’s read a book*, C immediately insists: *Two!* (*¿Do’?*), lifting her two thumbs, side by side. Occasionally, F or O picks out one book. Once it is read, C says: *Another one!* (*¿Oto!*) Once the second book is done with, she frequently requests yet one more — and is sometimes successful. We cannot know if C loses track of the quantity or realises she is requesting more than two books.

By contrast, C continues to use the ‘one, one, one’ scheme for three items until the age of 2;1. An example:

O is folding the laundry in piles on a bed. C comments on a pile with three T-shirts belonging to her: *One, one, one.*

From this point on, development in this domain branches out in different directions. We would not describe it as an explosion, since episodes involving the use of external representations of number do not become suddenly more frequent. Rather, it is a matter of diversification:

– Extending the use of the two-thumb gesture and/or the word *two* to describe collections made up of a wider variety of items (e.g., a present object and an object in the same category that is elsewhere — be it in the day-care centre or in the car).
– Using number words and/or gestures as labels for ages (two, one, and five).
– Extending the description ‘many’ to collections of three and four (e.g., to describe four chalks in disarray).
– Establishing contrasts between ‘two’, ‘one’ and ‘many’ (e.g., *No, not many! I said two!*).
– Stating approximate proportional relationships (e.g., *We have more time and we swim more*).
– Setting up relationships between numerical notations and quantities.
– Marking positions in a sequence with number words.
– Other enumerative activities.
– Distributing objects.

Episodes related to these strands occur at varying times. For some strands, a few or several episodes take place, whereas for others, a single episode was observed.

**DISCUSSION**

Although C communicates her numerical explorations, as well as producing and responding to *two* in daily conversation, and her social partners usually offer her their understanding followed by appropriate uptake, C’s representational constructive
activity in this domain almost exclusively serves the explicitation of knowledge content. In most of the observations C externalises a reading that she applies to objects (or displays of objects) in her environment (Observations 2, 3, 4, 5, 6, 7, 10, 13, and 14). The collections are often constructed by her — she sets up her own ‘reality’. She predicates properties of present, recalled, or anticipated displays (see Table 1). In doing so, C takes factuality for granted; she offers up her take, her reading, a ‘... representation (that) is used as a reflection of the state of the world’ (Dienes & Perner, 1999, p. 737). Her activity is essentially investigative rather than imitative, symbolic rather than empirical, conclusive rather than questioning. Her first steps into the world of mathematics are astonishingly smooth and structured. The clutter of transversal group studies is swept away.

In 1882, Preyer pointed out that no number concept or number treatment could exist if the infant or toddler did not grasp object permanence and did not possess some method for comparing objects (Preyer & Ekhardt, 1882/1990). We think that C’s activity rests on the representation of objects (object permanence, object concepts, object lists), the construction and consolidation of conceptual categories (see e.g., Mandler, 1992); in that sense, it can also be viewed as a symptom of budding classification, as it is based on an analysis of similarity and difference.

Her ‘one, one, one’ and ‘one, one’ schemes thematise the co-occurrence of unitary items that are physically similar and/or that she herself assigns to a particular category. Of course, C’s attention and underlying cognition are preformed to take some aspects of the environment into account, as without the capacity to see a tiled floor as consisting of identical or highly similar parts, without the certainty that elephants are elephants and not rabbits or chairs, her enumerative schemes would find no application (and would probably never have arisen). If C has a ‘number sense’ we did not see it, as we were not looking in the right place — we did not observe her handling of continuous quantity and did not ask her to judge, estimate, or discriminate the quantities shown in displays constructed by adults. We observed her acting in, and responding to, a rich environment. Nevertheless, it is striking that with one exception (Obs. 4), her visible activity is centred on collections of under 4 (see Table 1), i.e., the restricted universe where the ‘number sense’ is generally claimed to operate.

C’s constructions reminded us powerfully of the intuitionist school of mathematical theoreticians, such as Brouwer, who stressed the importance of units (see Hersh, 1997, p. 153 and ff., and von Glasersfeld, 1993). von Glasersfeld’s considerations about perception, unitary items and figural, spatial, and temporal patterns (von Glasersfeld, 1981, 1982, 1991, 1993, 1995; von Glasersfeld & Richards, 1983) loom large. C’s numerical exploration begins with the examination of a particular perceptual display that she suddenly notices — that speaks to her, calls up treatment — and that she subsequently returns to often as an exemplar that provides a prop (Obs. 2).

How does C pick out or choose the salient elements and orient her cognitive activity? Her activity, of course, is coconstructed in social interaction. The world of objects, the universe of people, their thinking as externalised in talk, meld to create a rich tapestry in which C herself participates and strives to participate more in.
‘Without language, no number!’ — see Hurford, 1987, p. 8.) We would be eager
to determine, for example, if her brother’s counting sequence and division of a
continuum (time) into units of measurement with vocal markers (see Obs. 1)
prompted C to explore another instantiation of this type of division or chunking,
collapsing it onto a perhaps (but only perhaps) simpler realm, that of things.

Before her second birthday, C has acquired the words one and two. She acquires
the word two by relating her own external representation to a conventional lexical
item by using yet another kind of representation. Her learning, as we observed it, is
doubly symbolic: to appropriate symbols she uses other symbols and sets up relations
between symbols or symbolic acts, necessarily using translation or paraphrase. We
may also note that she passes from using a logico-mathematical procedure in action
(one-to-one correspondence; although one might prefer to call it itemising or tagging)
to the setting up of a concept (two) which has a conventional meaning and formal
definition — roughly, describes a set of two entities belonging to the same category. At
this time, her concept of two does not entertain rich relations with other numerical
concepts or mathematical ideas. (We were not able to decide whether C, at age 1;10
realises that the difference between two and one is 1). The very fact that she discards
her enumerative schemes seems to indicate that their purpose was to extract and
note the unit and explore categories, and not to deal with one as part of a number
system. Her two is thus a protocardinal. A cardinal because it names a unique
collection type, a specific numerosity; proto- because it is not part of a system, and
the relations between the elements are fuzzy (at least to us). Possibly, the best way to
put it is that C has constructed a new unit — a ‘unit of units.’

Our interpretation of this matter rests in part on the observation of the boy L,
who did not use one-to-one correspondence at all. He was never seen to use such a
scheme, with the following exceptions: a) in action, when the pragmatic aim required
it, such as putting one toy car in each parking place, one fork for each (real) plate,
etc.; b) for ‘counting’, using many different number words. His enumerative lists
were of a different kind, produced at a later age, and focused on difference, with the
underlying category and itemising being assumed. For example: listing a group of
children with their proper names, struggling to express that each letter box belongs
to a different person, describing a collection as one red, one blue. L, to grasp
‘number’, needed to individuate elements, ‘tag’ them, probably to be able to keep
track of them mentally and work on composition. The unit he extracts is arithmetic, a
fallout of number templates and descriptions of small collections — a different
matter entirely. We mention the other child because we wish to emphasise that many
avenues are open for constructing number concepts. In mathematics, every entity or
regularity or rule is connected to some other element, as it is a human creation and a
symbolic structure. To unravel it, one may pull on one string or another.

We thus consider ourselves very fortunate to have been able to observe C at this
time of her life, and hope others may have not only enjoyed the attempt to describe
nature but will find food for thought.
The large majority of the observations concern discrete quantities and number forms: number words, gestures, and notations. The number forms, or external representations, were used by the children to describe or refer to: a) present or absent entities; b) entities that are grouped in some way, or joined, fixed together, or isolated; c) entities that are manipulable and transformable, or not; d) that are the focus of interaction, or not. Repeated actions, conventional division of a continuum (time), and numerals in the environment are also taken up.

Cuánta or cuánto is usually used to refer to large amounts of continuous quantities such as water or time, or states as mess or hunger.

Biblo seems to be a semantically based lexical creation, derived from biblioteca (library).

REFERENCES


FROM ONE TO TWO


ACKNOWLEDGEMENTS

We thank C and L for agreeing to seeing their younger selves displayed in the psychological literature. N. Scheuer counted on support by Universidad Nacional del Comahue (B-139), ANPCYT (06-1607), and CONICET (PIP 5663) and Ministerio de Educación y Ciencia de España (SEJ2006-15639 C02-O1) during the preparation of this chapter. We are grateful to Ernest von Glasersfeld for his thoughtful review of this chapter.

Nora Scheuer
*Centro Regional Universitario Bariloche,*
*Universidad Nacional del Comahue and CONICET*

Anne Sinclair
*Faculté de Psychologie et Sciences de l’Éducation,*
*Université de Genève*