Digital technologies permeate our lives. We use them to communicate, research, process, record, and for entertainment. They influence the way we interact in the world, the way we live. Digital technologies also offer the potential to transform the nature of the learning process in mathematics. The learning environment, the types of tasks learners can engage with, and the nature of that engagement differs from working in other environments. The Internet, for instance, presents greater scope for child-centered, inquiry-based learning. Dynamic geometry software and GoogleEarth offer interactive ways of exploring shape, position and space that is not possible with the pencil-and-paper medium.

This book provides insights into how mathematical understanding emerged for primary-aged children (5–13 years) when they investigated mathematical tasks through digital media. It considers learning theories that are frequently used in mathematics education, and situates a contemporary interpretive approach within those perspectives. A key purpose was to provide some practical tasks for teachers/teacher educators to incorporate digital technologies into their mathematics programmes, tasks that have been used successfully for learning.

This is a significant reference book for primary-school teacher education and a valuable resource for all schools teaching at that age.
Processing Mathematics Through Digital Technologies
Processing Mathematics Through Digital Technologies

The Primary Years

Nigel Calder
University of Waikato at Tauranga, New Zealand
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ACKNOWLEDGMENTS

Firstly, I would like to thanks the teachers and students involved in the ongoing research, whose voices resonate throughout this book. Their enthusiasm, the quality of their engagement, and their openness and honesty are greatly appreciated. I am indebted to their co-operation, and positive approach to the various studies undertaken.

I also wish to acknowledge the professional colleagues who have supported me. The ongoing critical discussions, the challenges we have posed and addressed, and the generosity and camaraderie of a range of educators across the world, but most especially in Australasia, the UK and France, has been highly influential in the thinking that underpins this book. In particular, Tony Brown, Jenny Young-Loveridge, Peter Grootenboer, and my Tauranga and Hamilton colleagues.

My heartfelt thanks to my wife, Margot, and my family, who have encouraged and supported me through the various research and writing journeys.
CHAPTER 1

AN EVOLVING UNDERSTANDING

INTRODUCTION

Do you use or have access to any item or device that includes an element of digital technology—something that perhaps has a digital technology as an input, or influences its use in some way? A phone, book, magazine, train ticket, plants you have grown from bought seeds or plants….? Such a redundant question! Digital technologies permeate our lives. We use them to communicate, research, process, record, and for entertainment. They influence the way we interact in the world, the way we live. How these influences shape our interactions and the way we think is the subject of much debate and conjecture. Digital technologies also offer the potential to transform the nature of the learning process. The learning environment and the manner in which learners engage in tasks differ, with consequential variations in both learner activity and dialogue compared to learning through other media. The Internet, for instance, presents greater scope for child-centered, inquiry-based learning. Dynamic geometry software (DGS) and Google Earth offer interactive ways of exploring shape, position and space that is not possible with the pencil-and-paper medium.

The intention of this book is to provide insights into the ways mathematical understanding evolves for primary-aged children when they investigate mathematical tasks through digital media. The book is in four sections. The first three chapters examine various theoretic perspectives. Chapter 2 is a discussion of the theoretical discourses that underpin approaches taken to understand the process of learning mathematics through digital technologies, and the inter-related influences that shape the understandings that emerge. Just how those learning experiences might be different from engagement through other media, such as pencil-and-paper, is considered, and how this might influence the students’ interactions, their learning trajectories, and their understanding. Central to this are the opportunities and constraints working in a digital environment might afford. These affordances are discussed in depth in Chapter 3 with reference to particular digital media, and also regarding general characteristics of working in digital environments. We need to consider what is different about the learning experience. Do digital technologies give alternative insights and allow the students to re-envisage the mathematical ideas in alternative ways? Do they open up new areas of mathematical investigation that would not otherwise be possible in primary-school mathematics classrooms, or give potential to transform the nature of primary-school mathematics?

A further aim is to provide some practical ways for teachers and teacher educators to incorporate digital technologies into their mathematics programmes. The next section consists of two chapters that introduce and discuss several
mathematical tasks and approaches to learning. These activities facilitate mathematical thinking and understanding at two distinct primary-school age groups: one, Chapter 4, aimed at the junior level (ages 5 to 8 years) and one, Chapter 5, at the senior level (ages 9 to 13 years). Here the New Zealand context is used but the tasks can just as easily be situated within school contexts in other countries. The introduction to these chapters will outline these commonalities and invite the reader to recognise similarities and contrasts with their own situation. As well, examples of tasks, and associated illustrative data, are threaded through the more theoretical chapters. The third section consists of three chapters that are concerned with influences on the learning process. They consider pedagogical elements and how the digital technologies interact with the preconceptions of the learner, the mathematical task, and any accompanying dialogue, to shape the learning process. This allows the learner opportunity to re-organise their mathematical understanding and explanations. The final section, discusses possible futures. These two chapters contemplate the implications for classroom practice: How might the use of digital technologies in mathematics transform the social organisation of the classroom? In what ways might communities of learning evolve beyond the constraints of school environs? Chapter 9 considers the potential for more child-centred, inquiry learning situations that explore realistic problems in an integrated manner. Chapter 10 casts a glance towards a possible future if hand-held web-based devices assume prominence and software and applications take on more formative, organic ways of developing through open-source processes.

Important too, is an examination of the key theoretical perspectives used to analyse the use of digital technologies in mathematics education- the lenses researchers utilise to focus on the way mathematical understanding emerges through digital media. Here, perspectives such as a socio-cultural viewpoint, with the medium acting as semiotic mediator, the instrumental approach, and humans-with-media are briefly considered. How they interact and gesture towards the moderate hermeneutic perspective to which the author subscribes is discussed, with any significant convergence or contrast examined.

This book is useful because it highlights the distinctive ways mathematical understanding develops through the use of digital technologies. It focuses on the primary-school setting, an area that doesn’t feature strongly in the literature, but has rich potential for using digital technologies in mathematics programmes. Primary-school teachers often bring creative approaches to the learning process and are often less constrained by rigid external influences, such as high-stakes assessment, than teachers in secondary school or tertiary settings. However, the book aims to bring clearer understanding of, and focus to, this aspect of teaching and learning for mathematics educators in a variety of settings.

But first, I begin with an illustration: An example of ten-year-old students investigating a mathematical task with spreadsheets. Spreadsheet software was used extensively throughout the ongoing study as it is freely available internationally, an aspect that addresses accessibility and possible equity issues. It is also software that I’ve frequently used with children. As you read the excerpt, consider whether it is a task that would be feasible in a classroom situation without
the availability of digital media to explore it. Also consider: Would the learning pathway have unfolded the way it did with pencil and paper? Would the mathematical thinking have been the same?

“LET’S JUST GO ON FOREVER!”

The data were produced when the group of 10-year-olds were investigating the task “Dividing one by the counting numbers”.

### Dividing 1 by the Counting Numbers

| When we divide 1 by 2, we get 0.5, a terminating decimal. |
| When we divide 1 by 3, we get 0.33333…, a recurring decimal. |
| Investigate which numbers, when we divide the number 1 by them, give terminating, and which give recurring decimals. |

*Figure 1. Dividing 1 by the counting numbers task.*

In the first case they negotiated to gain some initial familiarisation of the task.

*Sara:* One divided by one is one - it should be lower than one.

*Jay:* Try putting one divided by two, and that should be 0.5.

They then entered 1 to 5 in column A and \(=A1/1\) into cell B1 before using *fill down* in column B to get:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

The cells in column B were dividing the corresponding column A cell by one, hence producing the same sequence. The pupils were aware that the output should be less than one. This posed an immediate tension with their initial thoughts and fostered the resetting of their sub-goal. Sara’s preconceptions suggested one output, that it should be less than one, while at the micro level of the investigation the output was greater than one. The output didn’t reflect their expectation.
They continued:

*Sara:* Is it other numbers divided by one or one divided by other numbers?

*Jay:* Let’s recheck.

Jay entered =A1/4 and got the following output:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

*Jay:* Umm, we’re not going to get change… - we’ll have to change each one.

The girls explored other formula e.g., =A1/4 and =B1/(4+1), eventually settling on one that they thought modelled the situation.

*Jay:* Oh now I see =1/A1.

They generated the following output:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.33333…</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
</tr>
<tr>
<td>6</td>
<td>0.16161616…</td>
</tr>
<tr>
<td>7</td>
<td>0.1428514285…</td>
</tr>
<tr>
<td>8</td>
<td>0.125 etc.</td>
</tr>
</tbody>
</table>

The cells in column B now contained the output when one was divided by the number in the corresponding cell from column A. For example, one divided by one is one, one divided by two is 0.5, etcetera. They considered the numbers that produced terminating decimals and the consequences for their emerging informal conjecture. This engagement with the task influenced their overall perception of the situation. They re-interpreted their broader mathematical lens through the engagement with the task before reflecting on this output from this re-organised perspective.

*Sara:* So that’s the pattern. When the number doubles, it’s terminating. Like 1, 2, 4, 8 gives 1, 0.5, 0.25, 0.125.

*Jay:* So the answer is terminating and is in half lots. Let’s try that = 0.125/2; gives 0.0625-which is there.

[She finds it on the generated output in their worksheet].

The pupils suggested a new informal conjecture, and compared it to the existing data as they investigated the idea of doubling the numbers. The table gave them some other information however.
Jay: *1 divided by 5 goes 0.2, which is terminating too.*

[Long pause].

This created a tension with their most recent conjecture. After further exploration, they reshaped their conjecture, incorporating their earlier idea.

Sara: *If you take these numbers out they double and the answer halves.*

Jay: *That makes sense though, if you’re doubling one, the other must be half.*

Like 125 0.008; 250 0.004.

Sara: *What’s next. Let’s check 500.*

Jay: *Let’s just go on forever!*

The pupils generated a huge list of output, down to over 4260. They confirmed that when the divisor was doubled from 250 to 500, the quotient halved from 0.004 to 0.002, and likewise when the divisor doubled again to 1000, the quotient similarly halved to 0.001. Jay read off the table of output:

Jay: 500 0.002; 1000 0.001.

This indicated the relationship between the numbers that gave terminating decimals and the powers of ten. It led to a further informal conjecture, one articulated in visual terms:

Sara: *When you add a zero [to the divisor], a zero gets added after the point [decimal point].*

Sara was articulating an interpretation of the situation as envisaged through a particular school mathematics lens; for example, 5 gave an output of 0.2, 50 gave an output of 0.02, and 500 gave an output of 0.002. The following was also recorded on a piece of working paper, as a list of the numbers that produced terminating decimals:

1, 2, 5, 10, 20, 100, 1000

After recording two and five, it appeared they noticed that these were factors of ten and subsequently crossed them out. This observation also occurred with the twenty and one hundred, that is, they noticed that twenty was a factor of one hundred so crossed the twenty out. This interpretation was later verified with the pupils. They had made sense of, explored, and then generalised aspects of the investigation, culminating in the indication of a relatively complex notion of factors and understanding of a generalisation process.

So, with regards to using the spreadsheet to investigate the task:

1. Were the children involved in mathematical thinking?

Jot down your ideas as you think of them.

2. Would you consider it as a task to use in a classroom situation with ten-year-olds if there wasn’t digital media available to explore it?
3. Do you think the learning pathway, and the understanding that emerged, would have unfolded in the way they did without digital technology?

**Discuss your ideas with your neighbour.**

Without going into great depth at this stage (we will later) some of the aspects you might have considered are outlined below. Many of you will have others as well.

1. Mathematical thinking has been described as about using creative, critical, and meta-cognitive processes to make sense of information, experience, and ideas. It is when students learn to create models and predict outcomes, to conjecture, to justify and verify, and to seek patterns and generalisations (NZ Ministry of Education, 2007). Watson and Mason (2005) have described some of the acts of mathematical thinking as generalising, conjecturing, exemplifying, deleting, correcting, comparing, sorting, organizing, changing, varying, altering, explaining, justifying, convincing, and verifying. The children were doing all of these to some extent.

2. The difficulties with this might include too many computations, potential for inaccuracies, too time-consuming, hard to see the patterns, or that some examples would be too difficult for this level (e.g., 1÷3726) etc.

3. The children’s approach might not have included modelling (either visual or symbolic), looking for patterns or generalisation, exploring such a large amount of data, making connections, testing their informal theories so easily etc.

In Chapter 8, we will re-examine this excerpt through a moderate hermeneutic perspective to gain further insights into the ways learning emerges through digital pedagogical media. In the meantime, let us background this interpretation of the learning process by considering some perspectives on learning theory.

**LEARNING THEORIES IN MATHEMATICS**

This part of the chapter threads a theoretical trail; from broad social science philosophical beginnings, through the influence of various educative theories, leading to a brief discussion of how research framed by constructivist and socio-cultural discourses has influenced mathematics education. In particular, the acquisitional theoretical frame of Piaget, and Vygotsky’s socially-orientated one, are situated within literature associated with the ongoing evolution of learning theories in mathematics. A section investigating how those positions might be reconciled or enriched through a hermeneutic lens follows, with a discussion portraying my contention that a hermeneutic interpretive lens provides a productive filter for analysis.

We initially consider some of the literature around the more wide-ranging theories that helped inform my consideration of a learning theory to examine most productively the use of digital technologies in mathematics education. We never escape those socio-cultural discourses of tradition and authority that police the boundaries of more localised perspectives. They are interwoven and influential in the version of hermeneutics employed in this particular interpretation, to gain
insights, and to better understand, the ways new knowledge emerges. Beginning with this discussion of the literature associated with those broader influences, each subsequent section is informed by the previous one as the discussion threads a pathway from more expansive positions, through other formative influences, leading eventually to the moderate hermeneutic viewpoint to which I subscribe.

Moderate hermeneutics and the hermeneutic circle will be examined more closely in Chapter 3, but first several social science perspectives that inform these notions are considered.

Broader Views of Reality

By viewing mathematics education through an interpretive lens, there is acknowledgement of two fundamental aspects of interpretation: firstly, that there is an historically situated, socio-cultural space interpreters occupy from which they make their interpretation, and consequently, having interpreted phenomena, that space or position is transformed to some extent. From a poststructuralist viewpoint the interest is in investigating the historically situated nature of knowledge creation and its validation. From this perspective, it can be argued that the strategic purpose for that transformative practice is founded on the maintenance of power (Foucault, 1984; Philp, 1985). The first aspect, considering knowledge as being framed by historically situated discourses, validates a view of mathematics beyond that of being irrefutable fixed truth, to one of being an evolving process negotiated through a social consensus of language. This perspective “begins with the problem of unmediated access to a transparent mathematical reality, shifting the emphasis from the critical learner as the site of original presence, to a decentred relational complex process” (Walshaw, 2001, p. 28). It is this underpinning principle that guides the discussion that varying the pedagogical medium will lead to the evoking of alternative frames and underlying discourses, hence rendering the learning experiences and ensuing dialogue in a different manner. This allows space for the restructuring of mathematical understanding; for alternative ways of knowing. The following sections address this contention, and the evolution of this interpretive perspective of learning.

The notion that power has a particular role in what informs, and yet constrains mathematics education, indicates the potential ways in which mathematics education might be re-envisioned; how alternative understandings might be investigated, realised, and articulated (Klein, 2002). By recognising the constitutional influence of institutional power in what is defined as mathematics and mathematics education, the examination of that influence paves the way for the restructuring of the mathematics and mathematics education landscapes. Foucault (1984) argued that the social sciences have subverted the classical order of political rule based on sovereignty rights. This new regime of power is based on ‘norms’ of human behaviour. This establishment of ‘normality’ provides a framework for the vast area of ‘deviation from normality’. We delineate what is expected understanding at certain ages, as if contextual influences might be disregarded. These expected understandings might then be atomized into discrete
outcomes, further alienating the learner from authentic inquiry, and focusing educative practice on ‘helping’ individuals to attain those arbitrary norms. This indicates a version of mathematics education as a construction using incrementally more complex building blocks of content knowledge, rather than an evolving process of increasingly sophisticated thinking.

Language plays a central role in any ascribed version of mathematics education. Language is seen as a common factor in the analysis of social and individual meanings (Weedon, 1987) with its crucial role in the constitution of social reality making language critical in the contestation of meaning (Letts, 2006). One interpretation simply situates meaning as the negotiated interpretation of situations in which all phenomena are linguistic constructs. Knowledge and understanding only exist to the extent that they can be described. Mathematical knowledge emerges from linguistic, discursive activity. Those portrayals are a function of the space the participants occupy, their perspective, as much as the nature of the interactions. Those spaces evolve as participants position themselves according to the diverse influences that pervade their previous experience (MacLure, 2003). As they move in time through continuous space, there is a constant process of internalisation occurring as they connect with their environment and interact with it through language. A participant’s conceptualisation is not only shaped by this process but also shapes it. The participant and the process are symbiotically meshed. That mathematics and other traditional pure sciences are social constructs rather than descriptions of reality offers a model of learning based on the negotiation of meaning (Brown, 2001), or perhaps enculturation into a social practice. This unhinges rigid notions of mathematical truth and permits a perspective of mathematics as a flexible, contestable position (Klein, 2002). A corollary to this is that mathematics understanding can be envisioned by the way it is engaged, with the pedagogical medium crucial to the nature of that engagement.

This position, that mathematics understanding is a function of the way it is engaged, sanctions the contention that digital technologies acting as pedagogical media reorganise mathematical understanding. In particular, that investigating mathematical problems within a digital environment leads to alternative learning trajectories and understanding of the mathematics involved. Everything is framed and contextualised by the preconceptions and intentions of the user, yet interpreted through the lens of the learner’s prevailing discourse. This appears consistent with the contention that understanding evolves from the negotiation of meaning, and that learning is situated within the context of the experience.

The next section considers literature associated with other theoretical viewpoints that have influenced my position. This discussion will bridge various social science discourses leading to an emerging version of mathematics education and how mathematical understanding evolves.

Some Formative Theoretical Influences

Habermas (1976) tried to reconcile hermeneutics, the theory of interpretation, and a worldview where outside influences such as political forces, distort the
perspectives employed. Hermeneutics stresses that to understand human behaviour we have to interpret it’s meaning (Gadamer, 1976). We have to grasp the intentions and reasons people have for their activity. In the classroom setting we need to recognise and attribute the elemental causes of activity and dialogue, as well as describing and analyzing them. “Truth is the promise of a rational consensus” (Giddens, 1985, p. 130), so how can we differentiate truth from one based on power, or customs and traditions?

Habermas advocates the idea that power is a critical measure of existing interaction: it can highlight where consensus is based on tradition, power or coercion (Giddens, 1985). Other philosophers likewise flag the juxtaposition of perceived freedom of choice and the power hierarchies or traditions that actually shape those ‘freely’ made decisions. It is the influences of these discourses; of culture, society, and tradition, with all the historical, political, and power and submission voices with which they resonate, that frame the mathematical and media preconceptions that each learner brings to mathematical activity. Meanwhile, Gadamer also argued that understanding has two perspectives that frame its definition: firstly, as a holistic process reconciled by a multifarious framework, and secondly, as a dynamic process of encounter and response. He argued that it is preconceptions and prejudices that make the understanding possible in the first place (Gadamer, 1975).

It seems rational to argue that we interpret our approach to everything through that lens that is our present state, prejudices and all. Even when we experience quite cataclysmic events or have life-changing experiences, the catalyst or readiness for changes in understanding or perceptions, are embedded in our initial viewpoint. In mathematics education the learner brings a set of preconceptions and understandings to the new situation. These fashion the interpretations and hence the nature of the engagement in specific ways. In this particular discussion, we are concerned with the learner’s preconceptions of the pedagogical medium, and how these in conjunction with the affordances offered by the medium itself, promote distinct pathways in the learning process. Wittgenstein (1963) argued that the meaning of any utterance is a matter of its use and therefore, the understanding of any action or dialogue is dependent on the context in which it occurs. It would seem to follow from this argument that differences in context will affect understanding, even if the stimulus is constant. Allied to that is the contention that the pedagogical medium through which the action or dialogue is evoked, will also influence the nature of the understanding.

The difficulty is that the language used in that dialogue is not exclusively drawn from the learner’s perspective, but implicitly is coloured, or even shaped, by the viewpoints of previous users of the language and, in fact, society’s norms for the connotations of that language. An individual’s viewpoint can’t be seen as discrete from the communal perspective in which it was derived. Brown (1996) contends that as inhabitants of the world we are attempting to examine, we may describe our experience, yet these descriptions are imbued with societies’ preferred ways of saying things. Walkerdine (1988), in her investigation of the way young children learn mathematics, questioned whether the visual stimuli that manifest a particular
CHAPTER 1

connotation for one beholder, would produce the same nuances for others, given the context is consistent. Lerman (2001) described meanings as opaque rather than transparent; what they signify for the interpreter can’t be taken for granted. Meaning then is associated with more than the mathematical activity, but is framed by the individual mathematical and pedagogical discourses that constrain the associated interaction. While in this version of mathematics education the context clearly influenced this discourse, the discourse and sense making associated with it likewise shaped the context, and both are shaped through a larger more pervading lens of social practice.

Walker dine (1988) also maintained that the shift to the belief in the power of reason with, in mathematics education, its roots in the child-centred learning approach, is a shift in the perceived regulation of citizenship: a shift from the more overt, demonstrably authoritative power model and its inherent expectations, to one where the perception was outwardly of choice and freedom. She contends that the constrained and manipulated freedom that underpins the layers of choice was nonetheless regulatory. There is a sense of inevitability surrounding the way institutional discourses pervade individual preconceptions. To some extent this is an element of enculturation. It seems a logical extension, that an educational institution, or any institution, however loosely bound, will reflect in some form the political context in which it is set. Underpinned by a hierarchy of conceptual development, which depends on perception for cognition, Walkerdine contends that the references used in social discourse were not universal, “… but rather an aspect of the regulation of social practice which forms the daily life of young children” (Walkerdine, 1988, p. 11). New Zealand schools are no different, with political, regulatory and societal discourses holding sway in conjunction with mathematical and epistemological influences. While recognition of these determining features and the way they frame the learner’s perspectives is significant, the influence of the learning medium on the learner’s engagement, and the manner in which this relationship interacts with language to shape the learning trajectory, and hence the evolving understanding, is of greater concern to the purpose of this book.

This draws the discussion to phenomenology as it relates to mathematics learning. By envisaging mathematics as a social construct, as something arising in social activity, Brown (1994) reasoned that meanings of phenomena were located in particular contexts. He maintained that meanings are attributed to phenomena during the gaze of the individual through the lens of their personal perspectives. Understanding, in mathematics for the purpose of this discussion, emerges through the interpretations of phenomena, and while consensus of meaning evolves through language, no interpretation is ever final. The way that an object is encapsulated in the language of the subject, determines the interpretations that are evoked, but it requires a temporary fixation of time to allow interpretation to occur (Ricoeur, 1981). Hence, understanding in mathematics can be seen as the evolution of historically positioned meanings dependent on the spaces from which they are observed and the media through which they are encountered.
These broader theoretical positions gesture towards the interpretive perspective that I have privileged regarding the production of knowledge. They give validation to the fundamental premise of interpretation that there is an individual, historically situated, socio-cultural space the interpreter occupies from which they make their interpretation. The discussion also considered the assumptions that underpin the consequential notion, that having interpreted phenomena, that space or viewpoint is transformed to some extent.

Two of the principal constituent influences on learning theories in mathematics education will now be considered, leading to an examination of how the juxtaposition of their perspectives might enable them to be reconciled to some extent through an interpretive lens.

_Perspectives on Learning in Mathematics Education_

There are many theories of learning that are prevalent in the literature of learning in mathematics education, but two key perspectives underpin much of the discussion: that of Piaget, which is fundamentally an acquisitional perspective of learning, and Vygotsky’s, a participatory perspective. The following is a brief discussion of the two, and how they might be reconciled within a hermeneutic view of learning.

Piaget discussed the notions of assimilation and accommodation as learners interact with their immediate environment or situation. He described an ongoing process of “...assimilation of objects to schemes of action and accommodation of schemes of actions to objects” (Piaget, 1985, p.7). He portrayed these as occurring within a developmental framework of age-related stages that set parameters for intellectual growth. When a disturbance or tension emerges between their present understanding and a situation they encounter (i.e., cognitive conflict), the learner’s thinking evolves to balance that disequilibrium (i.e., resolve the conflict). He proposed a notion he called equilibration, a way to cognitive change via multiple disequilibrations and re-equilibrations (Piaget, 1985). He portrayed equilibration at several levels: the learner’s interaction with their world (as described above), interactions between sub-systems (schema) related to objects or actions, and equilibration between the sub-systems and their overall system of conceptual understanding. Learning can involve change in any of the three levels of equilibration and, he postulated, occurs when schema are re-organised through alteration (assimilation) or practice (accommodation). Inherent to this version, is the perception that any reorganisation will mean the subsequent re-equilibrations will be from fresh perspectives. The space the learner occupies will be different from that prior to reorganisation, and the multiple, ongoing engagements sustain the cognitive change.

Constructivism is a perspective underpinned by Piaget’s work, where the learner actively constructs the knowledge, and the learning is a process of adapting one’s view of the world as a result of this construction (von Glaserfeld, 1989). Learning, as per the constructivist version, can be construed as individual cognitive reorganisation (Lerman, 2001) or reflective abstraction employed as an apparatus.
for cognitive development (Battista, 1999). Meanwhile, Confrey and Kazak (2006) maintained that constructivism could effectively account for various classroom practices through a series of bridging theories such as Realistic Mathematics Education (Gravemeijer, 2002) or theories on mathematical thought that link learning processes inextricably with conceptualisation (Sfard, 1991). Social constructivism, with its negotiated meaning, depends on the commonalities the group traverses. While there is still a connection between activity and learning, it is the dialogue that arises as a result of the activity that leads to understanding (e.g., Schoenfeld, 1992).

While Piaget’s viewpoint is consistent with the notion of a personally constructed perspective of learning, Vygotsky saw learning as socially situated. He saw social participation evolving through transformative processes to become understanding. Vygotsky argued that individual knowing stems from relations between individuals, from human interaction (Vygotsky, 1978). As these relations are situated in particular times and places, the learning becomes socially and historically rooted. With varying, ongoing interpersonal experiences and consequential reflection, interpersonal events can over time become intrapersonal knowing, appearing to be increasingly abstract, but still tied to the series of events from which they are manifest. Vygotsky depicted the transformation of an interpersonal process into an intrapersonal one as the result of a sustained series of developmental events (Vygotsky, 1978).

As Vygotsky’s perspective of learning has cognition meshed to the particular situations of practice from which it emerges, the notion of abstract concepts that might be transposed across varying contexts is brought into question. Various commentators construe Vygotsky’s tenets in this manner, envisaging learning as located in co-participation in cultural practices (e.g., Cobb, 1994). Vygotsky’s articulation of the notion of tools as mediators and the semiotic mediation of language provide an historically situated, socio-cultural version of the process of understanding (Lerman, 2006). Research involving digital technologies in mathematics education frequently uses this theoretical frame to account for cognitive internalisation through the mediation of cultural tools (see Chapter Two). Participation in social interaction leads to understanding that may emerge from a range of situations, but Vygotsky perceived learning as the internalisation of social processes. In an educative sense, these social processes may be evoked by phenomena or some perturbation that might initiate dialogue or interaction.

The re-conceptualisation of mathematics learning theory from being an individual’s construction of understanding to that of enculturation, with mathematics perceived as a social construct, has evoked a pedagogical tension (Brown, 2001). The notion of enculturation, with the teacher as facilitator, places greater emphasis on the dialogue and therefore, the language in which the understanding is negotiated. An individual’s understanding is more deeply embedded in the collective sense made of the various mathematical stimuli and the relationships developed between students, and between students and the teacher, than merely the construction of meaning. The location of the learning also has greater significance.
It can be argued that the two viewpoints are complementary rather than mutually exclusive (e.g., Cobb, 1994). He advocated that the socio-cultural perspective informed theories of the conditions for the potentialities of learning, while theories developed from the constructivist viewpoint focused on what students learnt and the associated processes. It appears they are perhaps even more intimately entwined if one considers that an individual’s construction can only occur within a social framework. Confrey and Kazak (2006) likewise argued that learning in mathematics involves both activity and socio-cultural communication interacting in significant ways. They contend that neither influence is privileged, nor in fact can be separated, as we are simultaneously participants and observers in all enterprise, at all times.

Brown (2001) sought further clarity with examination of these issues from a contemporary hermeneutic perspective. From there he saw the formations of understanding evolving from both individual and collective interpretations of mathematical stimuli. These understandings develop through social activity and discourse, with all the historical, political, and cultural influences that such an interpretation implies. It follows that identical stimulus enacted upon in various pedagogical media will lead to different understandings no matter how subtly differentiated that might be. The differentiation is evident in the types of dialogue, both formative and explanatory; and the links made to other concepts that is, how the learner embeds the understanding in their existing schema, and how they might utilise these concepts or approaches in later mathematical investigation.

Research is also beginning to identify alternative areas of social-cultural theory that are emerging as current themes. The notion of identity, and how it behaves when social structures associated with a transformative process are in a state of flux, was considered to be crucial to the learning process (e.g., Lerman, 2006; Walshaw, 2008). Drawing on Boaler’s (2003) reference to the ‘dance of agency’ at the intersection of knowledge and thought, Lerman contends that the teacher’s task is to lay a mathematical identity among the sedimentation of personal identities. The work of earlier researchers underpin this approach (e.g., Lave & Wenger, 1991) who discussed learning in terms of a construction of identities, with the learner a participant in a socio-cultural world, and learning a process emerging from activity by specific people, in particular circumstances. One of these circumstances will be the learning environment with particular attention given to the pedagogical medium.

Other researchers (e.g., Lesh & Doerr, 2003) foresaw the emergence of modelling, as a key evolution of constructivism beyond individual cognitive composition. They reasoned that seeking generalities and consistencies within data and multiple-representations emphasises conjecture, promotes dialogue, and allows for the development of internal relations and meanings. They contend that this is of particular relevance to environments where digital media-based investigation is conducted, while acknowledging that the examination of model-based reasoning is still in its early stages.

Despite the fundamental cleavage between Piaget’s developmental theory and Vygotsky’s socially situated viewpoint of learning, when we attempt to reconcile
these seemingly polarised perspectives of the learning process through an interpretive lens, several commonalities emerge. Hermeneutics and a hermeneutic perspective to learning will now be considered with regards to the further insights into learning they might illuminate.

_Hermeneutics as a Window to the Learning Process_

Hermeneutics is understood as the theory of interpretation of meaning. While it was traditionally perceived in relation to the interpretation of text, Ricoeur (1981) rationalised spoken and written language as types of dialogue. It is not that they are the same, but that they have commonalities. There are a range of historical and philosophical positions that help situate hermeneutics, but a common strand is that in the process of interpretation no one facet exists in isolation. Each, whether author, text, listener, meaning, etc. has its own cultural, sociological, historical elements that fashion the interpretive process.

A moderate perspective of hermeneutics acknowledges the influences of time and space, as well as those of the conditioned prejudices that are embedded in language. The changing biases of various traditions are not past, but are operative and living in every social interaction, including oral and written (Gallagher, 1992). As the interpreter, we are constrained by our own language, but also by the language of the author, and the discourses that pervade both of these influences. “Understanding is always under the influence of history” (Gallagher, 1992, p. 90). An individual’s preconceptions or underlying discourse in a particular domain, influence their interpretation. This is similar to the idea of existing schema and assimilation from the Piagetian viewpoint, and echoes of Vygotsky’s recognition of the crucial role of social regulation and the social constitution of a body of mathematical knowledge (Berger, 2005). Further, Vygotsky argued that the child does not spontaneously develop ideas separately from their social context; that the child does not choose the meaning of his words, they are given to him through his conversations with adults (Vygotsky, 1986). A moderate hermeneutic perspective enables both viewpoints to be reconciled through the notion of discourse. The learner brings historically situated social, cultural, and political perspectives to the learning process.

Within the vast array and diversity of classroom experience there is nevertheless an interchange of learning, the scope and nature of which will differ with the situation. Although complex in its various manifestations, put simply the interchange of learning in the classroom situation is an interchange of interpretation. There is the interchange of interpretation, and thus learning, between teacher and pupil, pupil and pupil, and teacher and teacher that seems apparent, but also between the teacher and pupil with the pedagogical presentation. This model over-simplifies the full complexity of the classroom situation even given the social, cultural and historical discourses each bring to it. “The classroom is a curious and amorphous discursive space therefore-expanding and contracting under the pressures of different discourses that police its boundaries and construct its interiority in disparate ways” (MacLure, 2003, p. 11). Complexity exists within
perceived and demonstrable interpretation as well. The teacher’s understanding and the intention of their pedagogical presentation usually differ (Gallagher, 1992).

Although the interpretation of the presentation and the reconciliation of a consensus are indicative of the learning process, an echo of Piaget’s assimilation and accommodation, this does not necessarily happen. The pupil may be motivated to move to a different, unintended direction or be misled by, or examine or interrogate the presentation, but interpreting is implicit to the process that evolves. Similarly, the pedagogical medium evokes an interpretive response from the pupil. That the pedagogical medium might influence the interpretation and thus the understanding is central to this version of learning. The preconceptions or schemata guide the learner’s attempts to understand, but within that is a notion of constraint. The unshackling or broadening of these schemata is a key aspect of the learning process and it is the task of the teacher to create conditions that allow these preconceptions to be reshaped. The context of the learning needs to be set up on the basis of the child’s preconceptions, or the communication fails (Gallagher, 1992). Vygotsky likewise sees the role of the teacher, in the broad sense, as being central to the learning process, while Piaget acknowledges the teacher’s influence on the child’s individual construction of understanding.

Piaget suggested that disturbance results in cognitive change as the learner looks to re-establish a state of equilibrium, while central to Vygotsky’s theoretical position on learning is the learner’s participation in social processes. This contrasts with Piaget’s notion of the disturbance of existing structures, but there is place for dialogue and negotiation of consensus to emerge from tension evoked through interaction with new phenomena. The pedagogical medium might likewise bring about particular social responses. The hermeneutic circle combines notions of language and structure in emphasizing interpretation through the development of individual explanations (Gadamer, 1989). The learner develops explanations based on their interpretations of the phenomena. Their explanation then meets resistance from broader discourses, understanding evolves and the explanation alters.

Central to this interpretive process is the hermeneutic circle. This describes the process of the interpreter moving cyclically from the part to the whole, then back to the part and so forth, until some manner of resolution or consensus emerges. It is the circularity between present understanding and explanation, where the explanation gives rise to a change in perspective, which in turn evokes a new understanding. Within the learning context, the whole can be aligned with the various discourses or schema the learner brings to the situation, and the part with the specificity of the situation they confront (perhaps in the form of a particular learning activity). The learner’s engagement oscillates between their prevailing discourse and the activity.

Each iteration of the hermeneutic circle transforms the learner’s interpretation of the situation, while the pedagogical medium also influences their approach, and inevitably their interpretation and negotiation of consensus of meaning. This perspective enables Piaget’s position of ‘multiple disequilibria and equilibration’ to be viewed as a cyclical process alternating between disturbance and reconciliation with existing schema as understanding emerges. It also allows for a view of
learning as the transformation of an interpersonal process into an interpersonal one that Vygotsky (1978) depicted as “… the result of a long series of developmental events” (p. 57). This ongoing series of events moves between the phenomenon and the dialogue it evokes, with each iteration of social interaction a transformative process that shifts the interpretative frame to a new space from which the phenomena is viewed. With each of these iterations their perspective alters, and as they re-engage with the activity from these fresh perspectives, their understanding evolves.

Chapter 2 gives a more in-depth account of hermeneutics, including an illustration of the hermeneutic circle, and positions it within this explanation for the emergence of understanding. It also situates this within other prominent theoretical perspectives - those frequently utilised in the examination of the influence of digital technologies on the learning of mathematics.
CHAPTER 2

WAYS OF KNOWING

PRODUCTIVE FILTERS

This chapter describes and illustrates a particular version of the hermeneutic process used to examine the nature of the learning when mathematical tasks are engaged through digital technologies. Hermeneutics lends its title to a spread of theoretical perspectives, from a conservative position that endeavours to decipher literal interpretations of written text, through to a radical one that is inherently suspicious of language and opposes the transformative process of interpretation, seeing it more as a play of possible meanings. This chapter unfolds a moderate perspective, and uses data to illustrate how this view can be used productively to provide informative insights into the learning process through digital media. It begins with a brief traverse of key theoretical perspectives currently employed in the investigation of mathematics learning involving digital technologies. What is the nature of the interpretations they elicit? How might they inform each other and relate to the moderate hermeneutic perspective? A more fulsome account of the moderate hermeneutic position is outlined here, with an accompanying rationale and illustration.

A Socio-Cultural Perspective

A socio-cultural perspective of learning has various manifestations, but each has its embryonic beginnings in the work of Vygotsky. For Vygotsky (1994), communication was a cultural tool and communication was central to learning. Learning mathematics would involve participation in established mathematics cultural practices with students developing mathematical meanings as they learned to explain and justify their thinking (Steele, 2001). Vygotsky saw learning as socially situated, with understanding emerging from social participation. A Vygotskian perspective contends that individual understanding evolves from human interaction (Vygotsky, 1978). As the relationships in this interaction are positioned in particular times and places, the learning becomes socially and historically situated. These varying, ongoing interpersonal experiences coupled with associated reflection, can over time become personal knowing, seemingly becoming increasingly abstract, but still tied to the series of events from which they are manifest.

As this view of learning has cognition tied to particular situations of practice, it challenges the notion of constructed, abstract concepts that might be transported intact into a range of situations. Meanwhile, the Vygotskian perspective perceives tools as symbolic mediators that together with the semiotic mediation of language provide a socio-cultural version of the process of learning (Lerman, 2006).
CHAPTER 2

Research involving the use of digital technologies in mathematics education often utilise this frame to account for the internalisation of cognitive processes through the mediation of cultural tools (e.g., Marriotti, 2006). Participation in social interaction leading to knowing might be direct, or from observational viewpoints, or from internalised conversation (individual thinking), but Vygotsky perceived learning as the internalisation of social processes. In a similar manner, the objectification of understanding can be perceived as being underpinned by the interplay of typological meaning (language) and topological meaning (visual figures and motor gestures) (Radford, Bardini, & Sabena, 2007).

Moderate hermeneutics is likewise concerned with the socio-cultural influences that situate the learning experience and recognises the place of language in the learning process. It also contests the notion of concepts as absolute truths, but instead views them as evolving formative notions. We will return to these elements of moderate hermeneutics later in the chapter.

The Instrumental Approach

The instrumental perspective on tool use (Verillon & Rabardel, 1995) is frequently utilised in seeking clarification of the ways digital technologies and the learners interact, and the consequences of this interaction for learning. While acknowledging that the tool shapes the way the learner might engage in the mathematical task, the tool is not a transformative entity per se. They contend that the instrument does not exist interactively until the user has appropriated it into her activity. The notion of instrument includes the procedures and schemes that the learner develops through using the tool. These schemata guide the way the tool is used and the enhancement of the learner’s thinking (Kieran & Drijvers, 2006). This is complicit with the notion of instrumental genesis (e.g., Jackiw & Sinclair, 2006; Mackrell, 2006; Strasser, 2006) and its differentiation of an artifact and an instrument. In this, the digital technology, behaving as a cognitive tool, is seen as an extension of the mind. In this version of tool use, the instrument is more than an object, but encompasses the techniques and individual mental schemes that evolve through the use of the tool and social interaction. These guide both the way the tool is used and the user’s thinking. Instrumental genesis is the process that describes this transition from an artifact to an instrument. When mathematics is engaged through technology, mathematical knowledge is inextricably meshed with the knowledge about how the tool might be used (Laborde, Kynigos, Hollebrands, & Strasser, 2006).

The relationship between the user and the instrument is mutually influential. The affordances of the digital pedagogical medium constrain the way the user might interact with the mathematical task (see affordances, Chapter Three), while the preconceptions and understandings of the user guide the manner in which the instrument is used. Both the use of the instrument and the understanding of the user evolve in conjunction with each other, through iterations of interaction and reflection. This resonates with the moderate hermeneutic view, where understanding is seen as evolving through cycles of interaction and reflection as
the learner alternatively attends to the mathematical task through their preconceptions, their underlying discourses in mathematics and the digital medium. These interactions, and the associated language and reflection, in turn modify their perspective. The opportunities the media afford and the user’s interpretations likewise evolve through iterations of interaction and reflection.

**Humans-with-media**

The notion of humans-with-media is another theoretical position used to give insights into the ways understanding emerges through digital technologies (Borba and Villarreal, 2005). Humans-with-media describes a process where collectives of learners, media (in various, often collaborating forms) and other environmental aspects (e.g., other humans or technologies) interact with mathematical phenomena. They utilised a Tikhomirov perspective (1981) that claims the computer plays a mediating role in the reorganisation of thinking, and thus understanding. This mediating role is comparable, but not the same as Vygotsky’s conceptualisation that language mediates thinking. Borba and Villarreal (2005) saw understanding emerging from the reconciliation of re-engagements of the collectives of learners, media, and environmental aspects with the mathematical activity. They viewed these collectives in a dynamic way where the collective influences the approach to the mathematical phenomena, and is itself transformed by that engagement.

The humans-with-media collective is the collaboration from which the mathematical discourse in a particular domain emerges. It provides the lens through which the mathematical task is engaged. The engagement with the task, and the tension or opportunities this evokes, reorganise the thinking through the ensuing dialogue and action. From a hermeneutic perspective, what they say and what they do (Ricoeur, 1981). This transforms the learner’s perspective and hence the humans-with-media collective.

Learning experiences with digital technology, and the interfacing of these experiences with other media, reorganised learners’ thinking and transformed different human-with-media collectives (Borba & Villarreal, 2005). Borba and Villarreal saw this transformation emerging in a recursive way. As each engagement reorganised the mathematical thinking, and initiated a fresh perspective, this in turn transformed the nature of subsequent engagements with the task. This suggests that the process is ongoing, which also echoes the hermeneutic circle. Borba and Villarreal’s (2005) observation, that this process is recursive, is also indicative of the cyclical process of the hermeneutic circle, as the learner oscillates between the part (the mathematical phenomenon) and the whole (humans-with-media collective). The subsequent co-ordination with other media, including oral dialogue, that reorganises the thinking leads to this transformation of perspective. Implicit is the contention that this engagement and reorganisation of thinking is ongoing and iterative, at least until some consensus is reached.

Borba and Villarreal (2005) contend that the medium is significant in the reorganisation of thinking and, as a consequence, learners’ understanding. They argue that because of the sometimes unpredictable nature of the learner’s
interpretive perspective, “media, therefore, condition the way one may think, but
do not determine the way one thinks” (p. 16). The digital technology influenced the
engagement and ensuing dialogue in particular ways, leading to alternative
understanding and a modified perspective. The learner through self-reflection,
dialogue with others, or a combination of both, then resets their investigative sub-
goal and reengages with the task from the newly situated perspective. The iterative
process of the hermeneutic circle and the ensuing evolution of understanding
resonate with the humans-with-media notion and the corresponding reorganisation
of mathematical thinking.

MODERATE HERMENEUTICS

What Is Moderate Hermeneutics?

All hermeneutics theory is concerned with the interpretation of meaning. While
each of the various versions of hermeneutic theory (e.g., conservative, moderate,
radical, critical) can be rationalised with corresponding social science theoretical
discourses, the moderate hermeneutic perspective seems to resonate most
elocently with both my personal philosophical perspective and that of others,
whose various mathematical research informs this book. Gallagher (1992) contends
that each of these versions of hermeneutics, in different but complementary ways,
might present profound understandings of educational theory:

If education involves understanding and interpretation; if formal educational
practice is guided by the use of texts and commentary, reading and writing; if
linguistic understanding and communication are essential to educational
institutions; if educational experience is a temporal process involving fixed
expressions of life and the transmission or critique of traditions; if, in effect,
education is a human enterprise, then hermeneutics, which claims all of these as its
subject matter, holds out the promise of providing a deeper understanding of the
educational process (p. 24).

In the educational context we need to consider whether the objective is for the
learner to reproduce the meaning of the teacher/text (and if this is possible) or
whether the objective of the teacher/text intervention is to facilitate the learner’s
unique interpretation? Interpretation is not just determined by the lens through
which the interpreter views the phenomenon, but also by where their lens is
situated. The positions they occupy at various junctures have cultural, social,
political, and economic contexts that permeate their interpretation. These
contexts might be orchestrated; for example, with curriculum changes, or
manifest in a more natural, organic way; for example, the evolution of customs.
We need to consider to what extent these contextual influences are reproduced in
the understanding, a process that maintains domination, and to what extent they
are transformed, a process that evokes or has potential for emancipation.
Similarly, in the educational context, can the reflective process transcend or
transform these political and authoritative influences? Our response to this will
be guided to some extent by our approach to educational theory. A critical
approach to education will maintain that the power of reflection has the potential to fragment structures of power and authority in educational processes and institutions. Conversely, approaches consistent with moderate hermeneutics hold that structures of power and authority inevitably underpin educational experience (Gallagher, 1992).

Returning to the version of learning used earlier, we ascribed to the following view of understanding: that ‘concepts’ are not fixed realities we peel the outer layer from revealing their entirety, but more elusive, formative processes that become further enriched as the learners use their temporary fixes to view events from fresh, ever evolving perspectives. In essence, the mathematical task, the pedagogical medium, the preconceptions of the learners, and the dialogue evoked are inextricably linked. It is from their relationship with the learner that understanding emerges. This understanding is their interpretation of the situation through those various filters. Understanding emerges from cycles of interpretation, but this is forever in transition: there may always be another interpretation made from the modified stance. A moderate hermeneutic discourse provided a productive filter for analysing this version of learning.

With regards to digital technologies, these cycles of interpretation may evolve in alternative ways hence understanding will emerge differently when framed by alternative media. For instance, consider how the learning experience and the understanding might differ through groups working in the learning workspace below, as compared to an individual working in an exercise book.

Figure 2. Scratch software workspace.
An individual’s preconceptions frame their interpretation. A moderate hermeneutic perspective acknowledges the position of underlying discourses in the notion of preconception. The learner brings historically situated social, cultural, and political perspectives to the learning process. Within the diverse range of classroom experiences the process of learning will inevitably involve communication, with an interchange of interpretation. To have communication, things have to pass back and forth between participants with various personal perspectives. Brown (2001) discusses the spaces that emerge in conversation or activity, and considers them as gaps in which individual interpretations might be made. Learning becomes an exchange of narratives between interpretations of the world and existing explanations. The nature of the learning medium might likewise evoke particular social responses. The learner develops explanations based on their interpretations of the mathematical phenomena engaged through particular learning media. The hermeneutic circle combines notions of language and activity in emphasizing the place of interpretation in the evolution of understanding. The next section further examines the notion of the hermeneutic circle.

The Hermeneutic Circle

A central principle to the hermeneutic process is the hermeneutic circle. This was originally perceived as the circularity of the interpretative process as the focus shifted from the parts to the whole to the parts until a unity or consensus of meaning emerges. It has been conceived as a constant modification of the fore-structures of experience (Gadamer, 1976), which might be either fulfilled or disappointed. With fulfilment, the evolving preconception would be reinforced and be maintained as an interpretive influence; if disappointed the fore-conception is re-envisioned, with each revision conditioning the understanding (Gallagher, 1992). Hirsch (1987), made use of psychology terminology in the application of his model of ‘corrigible schemata’ to the hermeneutic circle; schemata which he contends are radically modifiable and responsive to context. The notions of existing schema, and historically and socially situated discourses, both echo of the hermeneutic circle as the pupil oscillates between the various discourses or schemata they bring to the situation, and the specificity of the situation they confront; that is, they move from the whole to the part with understanding shifting with each iteration. The constant modification of the schema is what the process of interpretation involves. In terms of the hermeneutic circle the meaning of a part is understood only within the context of the whole; the whole is never given without an understanding of the parts. “Every revision of the schema involves a recasting of meaning” (Gallagher, 1992, p. 64). The learner’s attempts to understand are hinged to their preconceptions or schemata, but by implication there is also an element of constraint. The unshackling or reenvisaging of these schemata is central to the learning process and it is the task of the teacher to create conditions that allow these preconceptions to be reorganised. “If the context of the learning is not set up on the basis of the child’s preconceptions... the communication fails”
WAYS OF KNOWING

(Gallagher, 1992, p. 79). Piaget and Vygotsky likewise recognise the role of the teacher in the learning process.

With most of the situations described in this book, the participants oscillated between the discourses summoned by school mathematics, language, and other broader social influences, and the activity with which they were engaged. Not only was their understanding negotiated through these filters and that of conversation within their group, but with ‘conversation’ with the pedagogical medium of the spreadsheet. They moved between preconception and their immediate reality. “The circular, dialogical structure of the teacher-student communication is maintained by the difference between the fore-structure (schema) operating in the students comprehension and the fore-structure that conditions the pedagogical presentation” (Gallagher, 1992, p. 75). Each iteration of the hermeneutic circle transformed their interpretation of the situation, while the pedagogical medium also influenced their approach, and inevitably their interpretation and negotiation of consensus of meaning.

One aspect that features in the research findings and discussion in this book, with implications for teachers, is the notion of visual perturbance. When the learner enters some input while working in a digital medium, they make decisions drawn from their prevailing discourses in the associated mathematical and pedagogical domains – their preconceptions guide their investigative pathway. They also guide the output they anticipate from this engagement. This output is frequently of a visual nature. When the actual output differs from that which is expected a tension arises, the learner is often surprised. Hence, the learner seeks to position this unexpected visual output within their existing understanding. In this way it evokes curiosity and often stimulates mathematical thinking, conjecture, and discussion.

In this book, I have coined the term visual perturbance to describe this tension. The word derives from perturbation, but more strongly echoes the term disturbance. Disturbance points towards the unsettling, interrupting and shifting effects of the visual output on students and moves away from the more emotive, upsetting and agitated connotations of perturb. Disturbing the students’ position is a necessary and positive process if we wish to open up their investigative process through a digital medium. While it is possible to orchestrate a visual perturbance in other pedagogical media, it is in a digital environment that a visual perturbance most easily, and naturally, occurs. Within the context of this discussion, it has an influential effect on the evolution of any associated hermeneutic circle.

Ricoeur’s (1981) notion of the hermeneutic circle emphasises the interplay between understanding and the narrative framework within which this understanding is expressed discursively, and which helps to fix it. While these ‘fixes’ are temporary, they orientate the understanding that follows and the way this comes to be expressed. In seeing understanding as linguistically based, it is appropriate that student dialogue and comment will provide the source for the interpretations of their mathematical understanding, in the domains considered in the research. Ricoeur (1981) parallels the relationship between spoken and written discourse, with action and the sedimentation of history. “History is this quasi-‘thing’ on which human action leaves a ‘trace’, puts its mark” (Ricoeur, 1981, p. 209). In this case, the evolving history of the learner is a collaboration of their dialogue and the corresponding action.
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A hermeneutic viewpoint allows the incorporation of dialogue and actions, as the links between what was being said or written, and the participants’ investigative approach, were examined in terms of their interpretation of the mathematical phenomena (Calder & Brown, 2010). Student dialogue and output are hinged to the discourse that constituted its production and analysis. An illustrative excerpt will give insights into the ways understanding might emerge when the learner interacts and interprets through these various filters.

Illustration of the Hermeneutic Circle

The following excerpt illustrates how a hermeneutic circle models the process by which learners come to their understandings. It applied to a localised learning situation drawn from a study, which involved groups of students investigating the 101 times table activity (see Figure 3 below). It demonstrates how one group’s generalisations of the patterns, and their understanding, evolved through interpreting the situation from the perspective of the preconceptions that were brought forth by their underlying discourses in the associated domains. These interpretations were from the perspectives summoned by personal discourses related to school mathematics, language, the pedagogical medium, and other socio-cultural influences. They influenced the manner in which the students engaged with and then investigated the task, while the interaction with the task and subsequent reflection shifted their existing viewpoint, it repositioned their perspective. The students then re-engaged with the task from that modified perspective. It was from this cyclical oscillating between the part (the activity) and the whole (their prevailing mathematical discourse), with the associated ongoing interpretations, that their understanding emerged. The excerpt also indicated elements that emerged through the moderate hermeneutic gaze that will be more fully addressed in Chapters 6, 7, and 8, that is, the stimulation of sub-goals in the investigative process, the visual perturbances evoked by the actual visual output conflicting with the expected output, and the capacity of digital technologies to enhance mathematical modelling. These were specific instances of localised hermeneutic processes, but while individually identified, they were interwoven with each other. The participants dialogue and output were the data used to illustrate these emerging themes.

<table>
<thead>
<tr>
<th>101 times table</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investigate the pattern formed by the 101 times table by:</td>
</tr>
<tr>
<td>• Predicting what the answer will be when you multiply numbers by 101</td>
</tr>
<tr>
<td>• What if you try some 2 and 3 digit numbers? Are you still able to predict?</td>
</tr>
<tr>
<td>• Make some rules that help you predict when you have a 1, 2, or 3-digit number. Do they work?</td>
</tr>
<tr>
<td>• What if we used decimals?</td>
</tr>
</tbody>
</table>

Figure 3. 101 times table task.
They begin the task:

Clare: Investgate the pattern formed by the 101 times table. When you multiply numbers by 101.

Diane: Times tables - so we just go like 2 times that and 3 times that.

Their initial engagement and interpretations are filtered by their preconceptions associated with school mathematics. “Times table” is imbued with connotations for each of them drawn from their previous experiences. The linking of the term to “multiply numbers” and “2 times that and 3 times that …” brings to the fore interpretations of what the task might involve. These position their initial perspectives. Their preconceptions regarding the pedagogical medium were also influential. It was from the viewpoint evoked by these preconceptions that they engaged with the task.

Clare: Just try 2 first, so one then two in that cell. Now go down.

The monitor displayed:

<table>
<thead>
<tr>
<th>A</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td></td>
</tr>
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<td></td>
<td>2</td>
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<td></td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Diane: It’ll be 2 times, no 101 then, 202.

They entered the following:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>1</td>
<td>101</td>
</tr>
<tr>
<td>202</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Clare: Yeah, but couldn’t we just go times 2 or 101 times.

Diane: Yeah, just do that.

Clare: You go equals, 101 times 2. Then you click in there. Oh man we did it. Now what are we going to go up to?

The students understanding of the situation, the mathematical processes involved (e.g., number patterns), and the pedagogical medium have influenced their engagement with the task, and the dialogue this evoked. This interaction has shaped their underlying perspectives in these areas and they re-engaged with the task from these fresh perspectives.

They re-entered the data with a change to the format to give the following:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>1</td>
<td>101</td>
</tr>
<tr>
<td>101</td>
<td>2</td>
<td>202</td>
</tr>
</tbody>
</table>
Diane: What we did was, we got 101. We went into A1 then we typed in 101. Then we typed in B1, and then we typed in equals A1 then the times sign then two. Then we put enter and we dragged that little box down the side to the bottom to get all the answers. That gives you the answers when you multiply numbers by 101. We multiplied two by 101. You get 202.

Clare: So you get the number, zero, then the number again. The next thing is to try other numbers. Like two zero, twenty.

They articulated an informal conjecture for a generalised form of the pattern, based on the visual pattern revealed by the spreadsheet structure, in conjunction with other affordances of the medium (e.g., instant feedback), and their mathematical preconceptions. They investigated the situation further from this fresh perspective.

Diane: So if we do two-digit numbers can we still predict?
Clare: So we’ll do like ten times 101. That’s a thousand and ten.
Diane: Shall we try like 306.
Clare: No, we’ll try thirteen, an unlucky number. That’ll be 13, zero, 13.

They entered 13 then dragged down:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>13</td>
<td>1313</td>
</tr>
<tr>
<td>101</td>
<td>14</td>
<td>1414</td>
</tr>
<tr>
<td>101</td>
<td>15</td>
<td>1515</td>
</tr>
<tr>
<td>101</td>
<td>16</td>
<td>1616 etc</td>
</tr>
</tbody>
</table>

Diane: Wow!!
Clare: Cool.
Diane: Now putting our thinking caps on.

They had anticipated an outcome of 13, zero, 13 (13013) when 13 was entered, consistent with their emerging informal conjecture, yet the output was unexpected (1313). There was a difference between the expected and the actual output, initiating reflection and a reorientation of their thinking.

Clare: Make some rules that help you predict. That would be the answer you get.
Diane: Like the 101 times table. Like we’ve got pretty much the 101 times table up on our screen because we just did that.
Clare: We had the number by itself then we saw that it was the double. So with two-digits you get a double number. What if we had three-digit numbers?

Diane: Let's try 100. That should add two zeros. Yeah see. OK now. Now, copy down a bit.

<table>
<thead>
<tr>
<th>101</th>
<th>100</th>
<th>10100</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>101</td>
<td>10201</td>
</tr>
<tr>
<td>101</td>
<td>102</td>
<td>10302</td>
</tr>
<tr>
<td>101</td>
<td>103</td>
<td>10403</td>
</tr>
<tr>
<td>101</td>
<td>104</td>
<td>10504</td>
</tr>
<tr>
<td>101</td>
<td>105</td>
<td>10605</td>
</tr>
<tr>
<td>101</td>
<td>106</td>
<td>10706</td>
</tr>
<tr>
<td>101</td>
<td>107</td>
<td>10807</td>
</tr>
</tbody>
</table>

Clare: Wow, there's a pattern. You see you add one to the number like 102 becomes 103 then you add on the last two numbers [02, which makes the 103, 10302]. So 102 was transformed to 10302.

Their engagement with the task has evoked a shift in their interpretation of the situation. The alternating of their attention from the whole (their underlying perceptions) and the part (the task), as filtered by the pedagogical medium and their interaction, was modifying the viewpoint from which they engaged and how they approached the task. It was from their interpretations of this interplay of influences that their understanding was emerging. This cyclical oscillation from the part to the whole continued with their viewpoint refining with each iteration.

Diane: Yeah, it's like you add one to the hundred and sort of split the number. Try going further.

They dropped the columns down to 119 giving:

<table>
<thead>
<tr>
<th>101</th>
<th>108</th>
<th>10908</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>109</td>
<td>11009</td>
</tr>
<tr>
<td>101</td>
<td>110</td>
<td>11110</td>
</tr>
<tr>
<td>101</td>
<td>111</td>
<td>11211</td>
</tr>
<tr>
<td>101</td>
<td>112</td>
<td>11312</td>
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<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>101</td>
<td>118</td>
<td>11918</td>
</tr>
<tr>
<td>101</td>
<td>119</td>
<td>12019</td>
</tr>
</tbody>
</table>

Clare: You see the pattern carries on. It works.

Diane: Look, there's another pattern as you go down. The second and third digit go 1, 2, 3, up to 18, 19, 20 and the last two go 0, 1, 2, 3, up to 19. It's like you're counting on. Try a few more.
Clare: Right our rule is add one to the number then add on the last two digits. Like 123 goes 124 then 23 gets added on the end 12423 see.

Diane: OK, let’s try 200. That should be 20100

They enter 200, getting:

| 101 | 200 | 20200 |

Oh… it’s added on a 2 not a one.

This unexpected outcome evoked a tension with their emerging generalisation, instigating reflection and renegotiation of their perspective. The direction of their investigative process shifts slightly; they propose a new sub-goal or direction to their approach and investigate further.

Clare: Maybe it’s doubled it to get 202 then got the two zeros from multiplying by 100. Try another 200 one.

They enter 250 then 251 with the following output:

| 101 | 250 | 25250 |
| 101 | 251 | 25351 |

Diane: No it is adding two now-see 250 plus 2 is 252 then the 50 at the end [25250]. Where’s that 2 coming from? Is it cause it starts with 2 and the others started with 1 [the first digit is a two as compared to the earlier examples where the first digit was a one]. See if it adds three when we use 300s.

They enter in the following:

| 101 | 300 | 30300 |
| 101 | 350 | 35350 |

Diane: Yes! Now 351 should be 354 and 51, so 35451. Lets see.

They enter 351

| 101 | 351 | 35451 |

Clare: OK then will you add 4 for the 400s? Lets see.
They enter some numbers in the four hundreds getting the following output:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>400</td>
<td>40400</td>
</tr>
<tr>
<td>101</td>
<td>456</td>
<td>46056</td>
</tr>
<tr>
<td>101</td>
<td>499</td>
<td>50399</td>
</tr>
</tbody>
</table>

Clare: That last one's a bit weird, going up to a 5
Diane: It's adding 4 though. See, 499 plus 4 is 503 and then the 99 at the end. Now how do we put this. It adds the first number to the number then puts the last two digits at the end. We'll put some more 400s in to see. 490 should be 49490 and 491, 49591. Try.

They entered those two numbers and then dragged down to get the following:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>490</td>
<td>49490</td>
</tr>
<tr>
<td>101</td>
<td>491</td>
<td>49591</td>
</tr>
<tr>
<td>101</td>
<td>492</td>
<td>49692</td>
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<tr>
<td>101</td>
<td>493</td>
<td>49793</td>
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<tr>
<td>101</td>
<td>494</td>
<td>49894</td>
</tr>
<tr>
<td>101</td>
<td>495</td>
<td>49995</td>
</tr>
<tr>
<td>101</td>
<td>496</td>
<td>50096</td>
</tr>
</tbody>
</table>

Diane: Yeah, it's working all right.
Clare: Seems to be. What's next?
Diane: What if we use decimals?

The students have negotiated a lingering consensus of the situation: one borne of their evolving interpretations as they engaged the task from their preconceptions in the associated domains. The ensuing interaction and reflection evoked subsequent shifts in their perspective. They subsequently re-engaged with the task from these modifying perspectives. Each iteration of the hermeneutic circle transformed their interpretation of the situation, with the spreadsheet medium influential to their approach, interpretations, and inevitably their consensus of meaning. The mathematical understanding that emerged was inevitably a function of the pedagogical medium employed, in this case the spreadsheet, and the interplay of their interactions as framed by their underlying discourses.

The various discourses frame the learner's attempts to understand but implicit to this is the notion of constraint. The challenging of preconceptions in a critical or at least reflective manner is an aspect of the learning process with the teacher's task to create conditions that allow these preconceptions to be reshaped. These schemata function in the same way as Husserl's notion of horizon, "supplementing the missing profiles with a pattern of meaning, that is constructing a perceptual interpretation" (Gallagher, 1992, p. 63). If a precept of education is transformation, then implicit to this is the notion of moving from the known to the unknown, the familiar to the unfamiliar. Gallagher reasoned that it was the context of the familiar, from which we negotiate the understanding of the unfamiliar, with this
context provided by the operations of tradition through language. Learning about something unknown always involves a preconception of what the unknown could be, given our prior experience and our prevailing discourse. Our interpretation of phenomena is either challenged or is reconciled in terms of what we already know. Our particular lens tints our interpretation.

The situation we hold, our positional viewpoint, influences the sense we make of unfamiliar phenomena. Likewise the interpretations made by the participants, the teacher, the author, and the readers were influenced by the space they occupied at that particular juncture and might have varied in different times. The hermeneutic situation is a localised interpretation, with the interpretive practices evolving within the local context. A hermeneutic frame might be prescribed, but only to a local context with which it subsequently becomes tied. The layering of these local hermeneutic situations informs the macro position, but each retains specificity to its evolution. The mathematical activity with which students engage, the classroom culture, and the pedagogical medium through which they interact will all influence the nature of any transformation, the nature of any understanding that evolves. While the mathematical task is a key element of this version that privileges mathematical thinking in mathematics education, the medium through which the tasks are engaged is likewise critical to the learning trajectory that unfolds and hence the understanding that evolves.