This volume presents the “state-of-the-art” of Nordic research on mathematics education within four broadly defined areas:

- the study and design of mathematics teaching in classrooms
- the identity and education of mathematics teachers
- the use of new technology in mathematics education
- meanings and challenges of providing mathematical education to all citizens in modern societies.

It provides the reader with insights into research done not only by scholars from the Nordic countries (Denmark, Finland, Norway, Sweden and Iceland), but also by colleagues from the rest of Europe – and even other parts of the world.

While the principal research questions addressed are universal in nature, their investigation in concrete contexts will inevitably relate to more contingent issues and conditions. This book offers both an in-depth view into the reality of mathematics teaching in the settings studied by the authors, syntheses by world renowned scholars of current problems and methods within each of the four areas, and cross-links to studies done in different countries, as represented both by this book and by the wealth of referenced literature it draws upon. Each of the book’s four sections therefore provides rich material for studies within the corresponding areas, for the beginner as well as for the expert.

The chapters of the book result from the work of the fifth Nordic congress in research on mathematics education, which was held in Copenhagen in April 2008. It includes 32 full research papers, 8 agendas and reports from discussions in working groups, and 22 short communications.
Nordic Research in Mathematics Education
Nordic Research in Mathematics Education

Proceedings from NORMA08 in Copenhagen, April 21-April 25, 2008

Carl Winsløw
University of Copenhagen, Denmark
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1. NORDIC RESEARCH IN MATHEMATICS EDUCATION: FROM NORMA08 TO THE FUTURE

This volume presents the work of the fifth Nordic conferences in research on mathematics education, which was held in Copenhagen, April 21-25 2008. The history of the Nordic conferences seems to be, in several ways, characteristic for and also linked to the more general emergence of a Nordic research community within the field loosely termed “mathematics education”. Research activities depended for a long time on a few pioneers’ initiative and interests. In fact, it is mainly within the last decade that the field has substantially expanded (e.g. in terms of tenured positions, international publications etc.) and has been formally institutionalised (e.g. with chairs and graduate education programmes at the region’s major universities).

The first Nordic conference – entitled “Nordic conference on mathematics teaching” (Lahti, 1994) – was held essentially at the initiative of a pioneer in the field, Erkki Pekhonen. He mentions, in the preface to the proceedings (Pekhonen, 1994, 10), that about two years ago, we discussed informally on the need of such a conference in the Nordic countries, and consequently put up the programme committee for the NORMA-94 conference.

“We” – i.e. the committee – involved, besides Erkki Pekhonen: Jüri Afanasev (Estonia), Ole Björkqvist (Finland), Gard Brekke (Norway), Bengt Johansson (Sweden), Jarkko Leino (Finland), and Ole Skovsmose (Denmark). The proceedings contains 33 papers, most by authors from Finland and the Baltic countries (2 from Sweden, 1 from Denmark, 1 from Norway).

The following conferences were held in Kristiansand (1998), Kristiansstad (2001), and Trondheim (2005). In each case, informal conversation at or after the conferences led to a decision upon how to organise the next edition.

The fifth conference was no exception: given that Denmark had not yet undertaken the task, and various other circumstances, it fell upon me to initiate the preparation of NORMA-08, and first of all to set up the two committees without whose collaboration the conference would never have been realised:

– the scientific programme committee (SPC), consisting of myself as chair, Erkki Pelkonen (Finland), Guðbjörg Pálsdóttir (Iceland), Gunnar Gjone (Norway), Jeppe Skott (Denmark), Johan Lithner (Sweden), Lisen Häggblom (Finland), Madis Lepik (Estonia) Marit Johnsen Høines (Norway), Paola Valero (Denmark), Tine Wedege (Sweden);
– the local organising committee (LOC), consisting of Lisser Rye Ejersbo as chair, Morten Misfeldt and Mette Andresen (all from Denmark).
The programme committee mainly worked on formulating the programme structure and the themes, to select plenary speakers, to make other general and crucial decisions, and not least to review the many papers which were submitted in response to the call. Members of the local organising committee took part in the last task as well, but above all they worked out the infrastructure of the conference: website, registration and payments, venue, excursion and so on. I extend my sincere thanks to all committee members for a pleasant and efficient collaboration and, on behalf of all participants in the conference, for the hard and entirely benevolent work they put into making this conference possible. Also on behalf of all conference participants, it is a pleasure to thank our three student assistants, who worked tirelessly throughout the conference: Rikke Anthon, Nadja Ussingkær and Karl-Otto Markussen. Karl-Otto also did an excellent job as a technical editor of the contributions for this volume.

Among the major early decisions by the SPC was that the conference would centre on four themes, defined in the call as follows:

- **Theme A: Didactical design in mathematics education.** This includes all types of “controlled intervention” research into the processes of planning, delivering and evaluating concrete mathematics education. It also includes the problem of reproducibility of results from such interventions.

- **Theme B: Education and identity of mathematics teachers.** This includes research into teacher education programmes, teacher educators’ practices, and the relation between teacher education and the formation of teachers’ professional identity and competence as mathematics teachers.

- **Theme C: Technology in mathematics education.** This includes studies of the rationales, modes and effects of technology use in mathematics teaching and learning at all levels.

- **Theme D: Mathematics for all: why? what? when?** This includes studies of mathematical literacy, rationales for “general” mathematics education, and the challenges of socio-cultural diversity in mathematics education.

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The themes were also used to structure the programme, as shown in Table 1: each theme was opened with a plenary lecture by a leading expert in the field,
followed by one or two sessions of regular papers on the following day. In addition to plenary, regular and short papers within the themes, the conference included submitted theme groups (to work with a particular topic with one of the themes) and a plenary panel to conclude the conference.

The fruits of the conference are presented in this book: more than 60 papers (including all formats), whose distribution on countries and on the four conference themes is shown in Table 2. It is clear that some of the papers are more loosely connected to their theme than others, but nevertheless the four themes have been operational to structure both the conference and this book. As regards countries, the strong appearance of Norway is impressive and reflects a strong development of mathematics education research in this country over the past decade; it can also not be denied that the host country appears with relatively few contributions, which in part reflects a relatively low number of researchers and doctoral students.

Table 2. Number of papers by theme and first authors’ country

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The choice of the four themes reflects two concerns:

– To include the best of current research on mathematics education in the Nordic countries, as well as in our neighboring countries (who participated more actively than ever);
– To point to areas which can be assumed to gain importance and momentum in the future.

The future of research in mathematics education in the Nordic countries point in several directions, some of which seem to be well captured by the four themes:

– The term “Didactics of Mathematics” – which is closer to the word used in Nordic languages than the term “Mathematics Education” – reflects the close historic and current bonds of Nordic research to continental European schools, like French didactique and German Didaktik. In each of these, the importance of design oriented research, as treated within theme A, is matched by an emerging focus on such research also among Anglophone researchers.
A major justification for research on the Didactics of Mathematics is the need for scientific support of the teaching profession and in particular the education of mathematics teachers, as considered by Theme B. Teacher education is therefore a natural major focus of future research.

The introduction of computers (in a wide sense) in mathematical practice has, of course, also increasingly affected the teaching of mathematics. While the theme is certainly not new in didactical research, as Theme C reflects it, many problems remain unsolved and new keep coming up as technology evolves. This area is likely to keep gaining momentum and depth in the coming years.

Finally, research in didactics of mathematics should never become autistically confined to the mathematics classroom. It needs to articulate and investigate the functions of mathematical education in the broader contexts of general education and society at large. This venture, as reflected by Theme D, is certainly one that will need to be further developed as more and more people will need, and get, still more advanced and varied forms of mathematical education.

It is obvious that none of these strong tendencies are specifically Nordic as such, and that they may take on specific characteristics in the context of each Nordic country. This, of course, could justify specific emphases and methods. But in my opinion, the most important trait of current Nordic research in didactics of mathematics – or, more broadly, Nordic research on mathematics education – is the maverick approach one finds even within single institutions, drawing upon and contributing to a multitude of international communities, some of which are also directly represented through the authors of this book. The book, as such, makes for a journey into not just current Nordic research on mathematics education but also to works of some of our friends and colleagues from the rest of the world.

As described in the very last section of this book, an umbrella structure was established during NORMA-08, in order to coordinate some of the national organisations of mathematics education research, and in particular to formalise the arrangement of the Nordic conferences. The sixth is to be held in Iceland in 2011.

REFERENCES


Carl Winslow
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Theme A: *Didactical design in mathematics education*
2. DIDACTICAL DESIGN IN MATHEMATICS EDUCATION

This paper, which results from a lecture given at NORMA08, proposes a reflection on the idea of didactical design, using the notion of didactical engineering as a specific lens. It is structured into three parts. In the first and introductory part, we point out the increasing attention paid to design issues in mathematics education, and some of its internal and external sources. In the second and main part, we come to the notion of didactical engineering analyzing its emergence in French didactical research, its development from the early eighties until now, and its influence on didactical design at large. In the third part, we extend the reflection by considering the way design issues are approached in the anthropological theory of didactics.

INTRODUCTION

Didactical design was one of the four themes structuring the NORMA08 conference and it was defined in the following way: “This includes all types of “controlled intervention” research into the processes of planning, delivering and evaluating concrete mathematics education. It also includes the problem of reproducibility of results from such interventions”. The choice made by the organizers can be seen as an expression of the renewed interest for design issues in mathematics education, as explained below.

The context

Didactical design has always played an important role in the field of mathematics education, but it has not always been a major theme of theoretical interest in the community. This is no longer the case today, the increasing interest paid to design issues going along with a more global reflection on the value of the outcomes of didactic research, and on the relationships between research and practice as attested for instance by the choice of this theme for one of the four surveys prepared for the congress ICME-10 in 2004 (Sfard, 2005).

Many reasons contribute to such a phenomenon. Some of these reasons engage the relationships between the mathematics education community and the society at large. Our field is still a young field of research but it is certainly old enough for being questioned about the quality and usefulness of the results that it has produced. The role that mathematics plays in the economical development of our societies makes mathematics education a sensitive domain from that point of view. Moreover, we are today faced with international evaluations such as the program PISA from OECD. Through the “measure” of students’ mathematical literacy and competences that they
pretend to provide, these evaluations question from the outside the efficiency of our educational systems, make more visible on the international scene successes but also limitations and failures, and lead to question educational research about its potential for informing and guiding curricular decisions and policies.

The increasing interest in design issues also results from factors more internal to the development of the field itself. The progression of research has made more and more evident that research methodologies have to organize a relationship with the contingency respectful of the situational, institutional and cultural dimensions of learning and teaching processes. Of course, research methodologies can be respectful of these dimensions without involving controlled interventions, but it is more and more acknowledged that controlled interventions have an essential role to play: for exploring forms of life that naturalistic observations cannot provide, for testing the resistance and pertinence of our theoretical constructs, for thinking and evaluating how research can impact practice in an efficient way, the ultimate goal of didactical research.

Many recent publications attest this increased interest for design issues in mathematics education and we briefly refer to two of these in the next paragraph.

Two recent references

A text often quoted regarding design issues is (Burkhardt & Schoenfeld, 2003). In this paper, the authors suggest “that educational research and development should be restructured so as to be more useful to practitioners and policy-makers, allowing the latter to make better-informed, less speculative decisions that will improve practice more reliably”. Starting from the claim that “educational research does not often lead directly to practical advances, although it provides useful information, insights, and ideas for improvement”, they plaid for the development of the kind of “engineering research” existing in many other scientific fields and making their results useful for practice: “The research-based development of tools and processes for use by practitioners, common in other applied fields, is largely missing in education. Such “engineering research” is essential to building strong linkages between research-based insights and improved practice. It will also result in much higher incidence of robust evidenced-based recommendations for practice, helping policy makers to make informed decisions.”

The authors consider then six different models of relationships between research and practice observed in education. The last one is that of Design experiments. According to them, this practice was introduced in the early nineties and represents a significant attempt to conduct research in (experimental) practice, and to contribute to both research and practice. They describe it in the following way: “Instructional interventions are designed with explicit theoretical grounding. Data gathered before, during, and after the intervention serve purpose of theory testing. At the same time, they point to strengths and weaknesses of the intervention, informing its revision. Iterative cycles result in improvements in theory and in refinements in the intervention” and claim that design experiments represent a much-needed melding of research and practice, but also that they embody only the early stages of the
The second paper we will mention is (Cobb, 2007). This chapter deals with the multiplicity of theoretical frames existing in mathematics education, and the ways researchers can cope with this diversity. For structuring the discussion, Cobb proposes to compare theories according to two criteria: the way they approach the cognitive individual on the one hand, and their potential usefulness given our concerns and interests as mathematics educators on the other hand. Its treatment of this second criterion, as he explains “is premised on the argument that mathematics education can be productively viewed as a design science, the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes”. According to him, such designs can situate at very different levels, from the classroom, the school to the school district of even the level of a national system. He also adds that his argument must not be interpreted as an argument in favour of one particular research methodology such as design experiments. According to him, a diversity of methods from experimental or quasi-experimental designs to surveys and ethnographies can contribute to this collective enterprise and be adapted to the questions at stake. We cannot enter more in the details of his analysis, but want to point out, the central position that Cobb gives to design in the conception of the research field itself.

Such texts convincingly express the importance of design issues and the need for deeper reflection and work in that area. They also show that the reflection must take into account the diversity of roles which can be given to didactical design, from the fundamental role that it has when mathematics education is seen as a design science to an application role when didactical design is seen as a way for organizing the relationships between research and practice or, in another words, for developing educational actions inspired by research and incorporating its results.

A reflective look at approaches sensitive to design issues

Looking reflectively at didactical approaches which have given a crucial role to didactical design is from that perspective certainly helpful. This is our purpose in this paper, using as a specific lens the case of the French didactical culture and more especially the notion of didactical engineering that emerged in it. In the French didactical culture indeed, design has always been given a central role. This has been achieved first through the notion of “didactical engineering” attached to the Theory of didactical situations (TDS) (Brousseau, 1997), and more recently also through the notion of “study & research programme” attached to the Anthropological theory of didactic (ATD) (Chevallard, 2006). French didactics shares this characteristic with some other didactical cultures. The first coming to the mind is often the Realistic mathematics education culture (RME), which has become more and more influential worldwide along the years, but this is also the case of the Open approach method developed in Japan, which has been popularized at international level through the process of “lesson study”.

A reflective look at approaches sensitive to design issues

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Neither TDS nor ATD are evoked in the two articles I have just mentioned. I think nevertheless that the long-term development of these theories and their relationship to design, make them especially appropriate for supporting a reflection on didactical design. I will thus focus on them in the following, looking reflectively at their historical development for approaching and discussing design issues.

**DIDACTICAL ENGINEERING**

The notion of didactical engineering officially emerged in the early eighties. A first institutionalization of it took place at the second Summer school of didactic (Chevallard, 1982). At that time, didactical design is clearly asked to answer two different needs:

- How to take into account the complexity of classroom at a time research mainly rely on laboratory experiments or questionnaires?
- How to think the relationships between research and action on educational systems?

Two different terms are in fact introduced: *phénoméno-technique* for the research methodology or practice (a term borrowed from Bachelard), *didactical engineering* for research-based action on educational systems (with the associated metaphor of the teacher seen as an engineer). But the first term will not survive in the community and the expression didactical engineering will become the polysemic notion that it is now. Unfortunately, this will contribute to occult the fundamental difference between research and action.

*The main characteristics of didactical engineering (DI)*

Seen as a research or development practice, DI is clearly a practice of the controlled intervention type, and this intervention is theory-based. In that case, the theory is the TDS and this deeply affects the vision of design. I cannot enter into the very details but would like to stress some characteristics of TDS especially influential from this point of view:

- The central role given to the notion of situation: In the TDS, the central object is the situation. Learning is viewed as an adaptation process depending on the characteristics of the situations where it takes place. The theory aims at understanding these dependences and at developing conceptual and methodological tools for their optimization.
- The crucial attention paid to the epistemology of knowledge: To each body of mathematical knowledge, the theory tries to attach a fundamental situation or a small set of fundamental situations reflecting its epistemological essence. This means, a mathematical situation such as the mathematical knowledge aimed at is an optimal solution to the mathematical problem it raises.
- The importance given to the characteristics of the milieu of the situation and to the characteristics of the students’ interaction with this milieu, trying to ensure a productive a-didactical adaptation [1], that is to say an adaptation based on the mathematical problem at stake, not on the guidance and help of the teacher or on the reference to the didactic contract.
The distinction between three different functionalities of mathematics knowledge, for acting, expressing and communicating, or proving.

The vision of the teacher’s role as an actor of the situation: mainly organizing the relationships between the a-didactical and the didactical dimensions of the situation through the management of the devolution and institutionalization processes.

These characteristics explain the specific attention paid in DI to epistemological and mathematical analysis, to the precise design of tasks and organization of the milieu, but also the rather limited role given to the teacher beyond its essential contribution to devolution and institutionalization processes. They also explain the structure of design attached to DI, with its successive phases of preliminary analysis of an epistemological, cognitive and didactical nature; tightly intertwined design and a priori analysis; implementation; a posteriori analysis and validation. More especially, the theoretical basis of DI explains the importance given in it to the a priori analysis, and the rejection of usual validation processes based on the comparison between the pre and post characteristics of experimental and control groups, at the benefit of an internal comparison between the a priori analysis and the a posteriori analysis of classroom realizations. This internal validation process of course does not exclude the use of methodological tools such as pre-test and post-test, questionnaires and interviews.

From a theoretical point of view, it is also important to point out that, from the beginning, didactical engineering has productively combined affordances from the TDS with those of other theoretical approaches developed in close connection to it, such as the tool-object dialectics (DOO) due to Douady (1984).

The golden age of didactical engineering

As analyzed in (Artigue, 2002), the eighties can be seen as the golden age of DI in French didactics. It becomes a major research methodology and proves its usefulness for identifying crucial didactical phenomena and working them out. Fundamental constructs emerge and develop through research projects based on DI, such as for instance the paradoxes attached to the didactical contract, the notions of institutionalization, of didactical obsolescence and reproducibility, and the notion of didactical memory. Moreover, the affordances of the TDS and DOO including the potential offered by the interplay between settings and registers for didactical design are made evident by the development of substantial pieces of DI, from elementary school to university. These are experimented and proved to be locally successful in the experimental contexts where researchers test them. The long term and impressive constructions elaborated by Brousseau and Douady respectively in the number domain for elementary school are the paradigmatic examples most often quoted, but many other convincing constructions are produced and tested, as attested by the doctoral thesis defended or the articles published in the journal Recherches en Didactique des Mathématiques.

Nevertheless, as pointed out in the course devoted to this notion at the Summer school of didactics in 1989 (Artigue, 1992), the transmission of didactical designs resulting from research DI is quickly perceived as something problematic. DI
realizations disseminate through different channels: publications of the IREM\textsuperscript{s}, pre-service and in-service teacher formation, textbooks\ldots, but the results are quite deceptive. Textbooks only propose very partial and superficial transpositions of the original constructions. Realizations observed in classrooms show important distortions, denaturized reproductions. My personal research on the notion of reproducibility contributes to reveal the complexity of the problems at stake. Through its mathematization, I establish first that the implicit model of reproducibility conveyed by didactical research cannot work. I show then that classroom dynamic belong to the category of non linear complex dynamics where micro-decisions regularly have macro-impacts, and that regularities in these dynamics have to be looked for, not at the level of students’ trajectories and classroom stories but at the higher level of story structures. These results lead me to articulate a principle of uncertainty between internal and external reproducibility (that is to say between a reproducibility preserving the external dynamics and story of the situation, and a reproducibility preserving the meaning of the mathematical knowledge built) in the following terms: “\textit{Relations between internal and external reproducibility should be seen in terms of uncertainty relations. In other words, the demand for external reproducibility can only be met by sacrificing internal reproducibility (internal reproducibility being the aim).}” (p. 59).

Looking retrospectively at this period, there is no doubt that the relationships between theory and practice as regards didactical design are not under theoretical control. Progressively, researchers becomes aware of this situation. Many of them incriminate the distance between the researchers’ metacognitive representations regarding mathematics as a discipline, its teaching and learning and those developed by teachers, and try to elucidate these representations using constructs from social psychology (Robert & Robinet, 1998). Soon enough, the question nevertheless arises of the links existing between these representations and the decisions that the teachers take in their teaching, globally and locally. The difficulties met with the transmission of DI products become thus a source or research questions. These will substantially contribute to the research developments observed in the nineties.

\textit{Didactical engineering today}

The difficulties met with the transmission of DI realizations have shown the necessity of considering the teacher as a full actor of the didactical situation, of understanding better her contribution to classroom dynamics and its effects, as well as the rationale underlying the decisions she takes. A better understanding of teachers’ practices and of the determinants of these becomes thus a priority in the research agenda. Since the early nineties, this leads:

- to the development of less invasive research methodologies, and the increasing importance given to naturalistic observations carried out in ordinary classrooms;
- to the development of theoretical constructions inside TDS or tightly linked to it (such as the refinement of the notion of didactical contract, the evolution of the vertical structure associated with the notion of \textit{milieu}, new models of the
didactical action of the teacher (cf. Laborde & Perrin-Glorian 2005) for many illustrative examples), but also outside TDS such as for instance the ergonomic-didactic approach due to Robert and Rogalski (2002),

and to a substantial body of research on teachers’ practices impacting the vision of didactical design.

Today, DI is no longer a privileged research methodology but it is still a tool widely used, especially for exploring forms of life which cannot be observed in standard contexts. From this point of view, the case of ICT research can be seen as a paradigmatic case: most doctoral thesis developed in that area in France includes indeed a DI dimension. As a research methodology, DI has also evolved. When looking at recent research making use of this methodology, one can easily identify some invariants: the same epistemological and mathematical sensitivity, the same importance given to the design of tasks, to the organization of a milieu offering strong potential for didactic adaptation, the same global structure with the same importance given to the a priori analysis, and the same internal process of validation. But, what is also evident, is that the proposed constructions offer a more flexible and realistic vision of the sharing of the mathematical responsibilities in the classroom between teacher and students, and a more sophisticated vision of the teacher role. Beyond that, the migration of this methodology towards other disciplines or towards other mathematics education cultures (such as the Italian or the Nordic cultures) has led to original constructions. (Terisse, 2002) offers for instance interesting illustrations of the use of DI in the didactics of sports; the doctoral thesis by Maschietto (2002) and Falçade (Falçade, Laborde & Mariotti, 2007) show the kind of powerful DI construction which can result from the interaction between the TDS and the Theory of semiotic mediation.

Nevertheless, there is no doubt that DI is still a research tool. Transforming it into the development tool aimed at initially is still a problem to be addressed, both from a theoretical and practical point of view.

THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC AND DIDACTICAL DESIGN

ATD developed as a generalization of the theory of didactical transposition, which can be seen as a theory aiming at the understanding of the ecology of mathematics knowledge. In ATD, a fundamental claim is that knowledge emerges from human practices and is shaped by the institutions where these practices develop. Human practices are modelled through the notion of praxeology which, at the most elemental level, are defined as 4-uplets made of a type of task, a technique for solving this type of task, a discourse explaining and justifying the technique (technology), and a theory legitimating the technology itself. As was the case for the theory of didactical transposition, ATD first developed as a descriptive theory, as attested by the many papers and doctoral thesis relying on it and published in the nineties. In the last decade nevertheless, a design dimension and different constructs of interest for didactical design at large have entered the theory. This evolution is for instance clearly visible in the plenary lecture given by Chevallard at CERME4. In that lecture, criticizing what he calls the “monumentalistic” doctrine pervading contemporary school epistemology, he plaid for a new epistemology in the following terms:
“For every praxeology or praxeological ingredient chosen to be taught, the new epistemology should in the first place make clear that this ingredient is in no way given, or a pure echo of something out there, but a purposeful human construct. And it should consequently bring to the fore what is raisons d’être are, that is, what its reasons are to be here, in front of us, waiting to be studies, mastered, and rightly utilised for the purpose it was created to serve.” (Chevallard, 2006, p. 26)

The notions of “study & research activity” (SRA) and “study & research course” (SRC) have been developed for supporting this new epistemology. As explained by Chevallard:

“A S&R programme is essentially determined by the will to bring an answer, A, to some generating question, Q, but it is also determined by constraints imposed upon the study & research to be done by the existing curriculum – which, for example, will not allow the class to draw upon such and such advanced mathematical praxeology.” (ibid., p. 29).

Considering the way ATD shapes or can shape the vision of didactic design, it seems important to mention also two other constructs: the scale of levels of didactic codetermination and the dialectic of media and milieus [2]. The scale of levels of didactic codetermination certainly can help to make didactical design sensitive to the different kinds of constraints to which didactical action is submitted, from the civilization level to the subject level. The dialectic of media and milieus a priori appears as a powerful tool for taking in charge the way technological evolution radically changes our access to information, and the characteristics of the milieus we can interact with for developing knowledge.

In the recent years several researchers, in France and Spain mainly, have developed and tested didactical designs based on these constructions as attested for instance by the contributions presented at the second congress on the ATD which took place in Uzès, in October 2007. Can such realizations have more impact on practice than didactical engineering? And under what conditions? This is today an open question. The distance existing for instance between most S&R programmes presented at the congress just mentioned and the usual curricular organization leads nevertheless to have some doubts.

ATD and TDS

For years, many researchers have combined the affordances of ATD and TDS. Considering the new sensitivity of ATD for design issues, it becomes interesting to investigate the similarities and differences between the two approaches from this point of view. There is no place enough in this paper for developing a detailed analysis and we will just mention a few points:

There is no doubt that the two approaches share an equal sensitivity to epistemological issues: mathematics knowledge is developed for answering questions. Didactical design must make these questions visible and organize their study.

ATD seems to give more importance to the existing cultural answers to these questions. Thus the necessity of organizing a critical work on these cultural answers...
but also the acknowledgement of their necessary contribution to the milieus with which the students interact, the dialectics between media and milieus playing an essential role. From this point of view, the logic underlying the notion of fundamental situation in the TDS, and the relationships between the adidactic and the didactic situations shaped by the dual processes of devolution and institutionalization present evident differences with the ATD logic.

ATD offers a global structuration of design shaped by the 6 study moments, the full development of praxeologies, and the progression from point to local and regional praxeologies. Once more, the fundamental role played in the TDS by the distinction between the three different functionalities of knowledge (action – formulation – validation) shows that another global logic is at stake.

The differences we have pointed out do not mean that TDS and ATD affordances cannot be combined in didactical design in a productive way, but they show that each theory approaches and shapes didactical design in its own and specific way, and that these specificities and their possible didactic effects need to be understood.

CONCLUSION

Coming back to the general issues evoked in the introduction, didactical design has certainly a essential role to play in the establishment of productive links between research and practice but the reflection developed in this paper about didactical engineering shows us that making this effective requires considering development not just as a by-product of research. Successful development requires specific research and structures able to organize and evaluate the effects of progressive up-scaling. It cannot be planned independently of associated programs of professional development for teachers. It requires substantial funding, long-term projects and new partnerships. It cannot be the task of some individuals, neither let in the sole hands of researchers. It requires also a substantial change in research values, and a rupture with the top-down approach which still leads development too often. All of this questions the analogies and metaphors we use for approaching the relationships between research and practice, development and dissemination.

NOTES

1 The notions of adidactical and didactical situations are theoretical constructs. A real classroom situation is generally more adequately modelled as a succession of adidactical and didactical episodes, the teacher substantially contributing to the dynamics of the situation by acting on the milieu.

2 The term medium is defined by Chevallard as any social system pretending to inform some segment of the population or some group of people about the natural or social world. The term milieu has to be understood with the meaning giving to it in TDS.

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3. ADDITION AND SUBTRACTION OF NEGATIVE NUMBERS USING EXTENSIONS OF THE METAPHOR “ARITHMETIC AS MOTION ALONG A PATH”

Negative numbers make up a domain of mathematics which is often considered difficult to teach and difficult to understand. According to some researchers abstract mathematical concepts are understood through conceptual metaphors. Using the framework of conceptual metaphors developed by Lakoff and Núñes (1997, 2000) a theoretical analysis is made of one of the characteristic metaphors for this domain in order to see what entailments an extension of the metaphor has. The analysis elicits three shortcomings of such an extension that might influence students’ comprehension: loss of internal consistency, loss of external coherence and enforcement of properties or structure that are not part of the original source domain.

INTRODUCTION

“They [negative numbers] lie precisely between the obviously meaningful and the physically meaningless. Thus we talk about negative temperatures, but not about negative width” (Martínez, 2006, p. 6)

Recently some researchers have emphasized the important role of metaphors in learning and teaching mathematics (Frant et al., 2005; Lakoff & Núñes, 2000; Sfard, 1994; Williams & Wake, 2007). This was also the topic of working group 1 at the Fourth Congress of the European Society for Research in Mathematics Education 2005. One of the questions for further research posed in this working group was: What are the characteristic metaphors, in use or possible, for different domains of mathematics? (Parzysz et al., 2005). The aim of this paper is to make a theoretical analysis of one of the grounding metaphors identified by Lakoff and Núñes (2000) and some extensions imposed on it by teachers when dealing with negative numbers. The chosen metaphor is “Arithmetic as Motion Along a Path” as it is used when the number line or thermometer is the source domain of the metaphor. In textbooks and classrooms in Sweden these are common representations referred to through metaphorical reasoning when dealing with negative numbers.

THEORETICAL FRAMEWORK

“A metaphor is what ties a given idea to concepts with which a person is already familiar.” (Sfard, 1994)
In the theory of conceptual metaphors a metaphor is seen as a mapping from a source domain to a target domain (Kövecses, 2002; Lakoff & Johnson, 1980). If a representation or model [1] is to function as a source domain of a conceptual metaphor, the source domain must be well known. Properties of the target domain are then understood in terms of properties of the source domain. Most of the ‘metaphors we live by’ (cf. Lakoff & Johnson, 1980) are based on experiences that are products of human nature; physical experiences of our body, of the surrounding world or of interactions with other people. The target domain of a metaphor is never identical to the source domain. Metaphors make sense of our experiences by providing coherent structure between the two domains, highlighting some things and hiding others. The theory of conceptual metaphors implies that different metaphors are used to structure different aspects of a concept. A conceptual metaphor is a mapping from entities in one conceptual domain to corresponding entities in another conceptual domain. Concerning mathematics Lakoff and Nunes (2000) claim that understanding basic arithmetic requires conceptual metaphors with nonnumeric source domains and that “abstraction of higher mathematics is a consequence of the systematic layering of metaphor upon metaphor” (p. 47). Basic arithmetic, they argue, is understood through four grounding metaphors:
- Collection of objects, combining, taking away (numbers as collections of objects)
- Construction of objects, combining, decomposing (numbers as constructed objects)
- Measuring lengths, comparing (numbers as length of segments)
- Motion along a path (numbers as points on a line)

These metaphors are sufficient for us to understand arithmetic with natural numbers, but in order to extend the field of numbers to include zero, fractions and negative numbers the metaphors also need to be extended, or “stretched” (Lakoff & Núñes, 2000, p. 89). The authors go on describing a way of stretching the metaphors and blending several metaphors so that their entailments together create closure for arithmetic in the enlarged field of numbers. The analysis section of this paper will focus on the last of the grounding metaphors; “Arithmetic as Motion Along a Path”, and see what happens to it when it is extended to meet the requirements of negative numbers.

CONCERNING THE USE OF METAPHORS IN TEACHING NEGATIVE NUMBERS

Using the theory of conceptual metaphors (Lakoff & Núñes, 1997, 2000) one could say that understanding the number line as a conceptual metaphor for arithmetic is by treating the number line as a source domain of the metaphor ‘Arithmetic as Motion Along a Path’. To make use of this metaphor it is necessary that the students are well acquainted with the source domain and the mappings between the source and the target domains. “A metaphor can serve as a vehicle for understanding a concept only by virtue of its experiential basis.” (Lakoff & Johnson, 1980, p. 18).

A metaphor can be activated by a representation of some kind. Representing numbers by a drawn number line or an elevator going up and down are different ways of representing numbers using the “Motion Along a Path” metaphor. In some ways the number line and the elevator are different source domains, but in some
ADDITION AND SUBTRACTION OF NEGATIVE NUMBERS

ways they are similar since both refer to our experiences of motion along a path. Mathematics teachers often use visual representations, such as manipulatives or drawings, here referred to as representations or models as source domains of metaphors. Thus, the number line is not simply a representation of numbers, but rather a representation of numbers through the metaphor “Motion Along a Path”. The model will highlight properties of numbers that are similar to properties of moving along a path, whereas other properties will be out of focus. When a representation is visualised by a person without being visual per se, it can be referred to as a mental model.

There are many pedagogical models and representations in use to illustrate negative numbers, the most common ones in Sweden being the number line (often as a thermometer) or the use of money (gaining and borrowing). These models function as source domains of different grounding metaphors. Researchers argue about whether the use of such models help student understanding or not.

According to Linchevski & Williams (1999) some researchers argue against using models for negative numbers, whereas they themselves claim that subtraction with negative numbers can be understood through the use of models. Whether the use of several different models simultaneously will confuse or help the student is also a matter of consideration (Ball, 1993; Kilborn, 1979). Gallardo (1995) suggests teaching negative numbers using discrete models, where whole numbers represent objects of an opposing nature, rather than using the number line. Frant et al. (2005) show that teachers sometimes use metaphors without being aware of it, and sometimes are aware of using them, but not of the possible difficulties involved. The importance of being aware of the limitations of a model was shown by Kilhamn (in press). The following section will reveal some of the limitations of the number line as a model for negative numbers.

ANALYSIS

The following section will show how the number line and the thermometer can serve as a source domain for addition and subtraction involving negative numbers through the “Arithmetic as Motion Along a Path” metaphor, and what happens to the metaphor when it is extended. Analysing the metaphor in the manner attempted in this paper was first done by Lakoff and Núñes (2000, p. 72–73) and further elaborated by Chiu (2001, p. 118–119)

The line, drawn horizontally as a number line or time line, or vertically as a thermometer, is seen as a path and addition and subtraction are seen as motions along this path. See table 1.

So far the mapping is coherent with our embodied experiences. There are, however, aspects of the target domain with no corresponding aspects in the source domain. What teachers and textbooks tend to do in these cases is to extend the metaphor by reversing the direction of the metaphor and turn the target domain (M) into the source domain (S).” The teachers’ source domain is mathematics and the target is daily life because they try to think of a common space to communicate
with the students” (Frant et al., 2005, p. 90). Situations are made up in domain S to fit the situations in domain M and thus create external coherence between domain S and M. Unfortunately many of these made up situations lack experiential basis and seem quite ‘unnatural’, and it is difficult to keep the domain internally consistent.

Table 1. Mapping of the number line/thermometer metaphor

<table>
<thead>
<tr>
<th>Source domain (S)</th>
<th>Target domain (M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centre point on the line (here called the origin)</td>
<td>zero</td>
</tr>
<tr>
<td>A point on the line relative to the origin</td>
<td>a number</td>
</tr>
<tr>
<td>A point to the right or above the origin</td>
<td>a positive number</td>
</tr>
<tr>
<td>A point on the line to the left or below the origin</td>
<td>a negative number</td>
</tr>
<tr>
<td>A distance on the line</td>
<td>absolute value of a number</td>
</tr>
<tr>
<td>Motion to the right or upwards</td>
<td>addition</td>
</tr>
<tr>
<td>Motion to the left or downwards</td>
<td>subtraction</td>
</tr>
</tbody>
</table>

Let us look at the expression $a + b = c$. In domain M; a, b and c are all numbers. In domain S (the metaphor of Arithmetic as Motion Along a Path) only a and c are understood as points on the line, whereas b is not a point on the line but a number of steps to move; it is seen in connection with the operation sign. We now have two different types of numbers. In this source domain there is a great difference between a point on a line and a motion along the line. The motion is always positive or zero. You either move or you don’t move. There is no such thing as anti-motion. Motion can be in different directions, but the direction is settled by the operation sign. Now assume that b is a negative number. What would that be in domain S when the idea of moving a negative number of steps is not part of our experiential basis? There are two common ways of solving this by extending the metaphor. The first way is to impose direction and start speaking of motion forward and backward relative to the person moving. The default direction is to face right. In order to achieve consistency we need to make some changes to the mapping. The metaphor now has to involve somebody (or something) with front and back. Since the temperature scale or the passing of time lack front and back they are of no use in this mapping. See table 2.

Notice how complicated the structure of domain S becomes. When teachers use this mapping to explain addition and subtraction in their classroom they often get confused and lose track of which way they are facing and moving (Kilborn, 1979). The metaphor also comes into conflict with other metaphors that are deeply rooted in our culture such as ‘positive is up and forwards’, ‘negative is down and backwards’. ‘Adding on is up’ (we grow up, we build up etc). ‘Right is forward’ (writing flows from left to right). The overall external coherent system of metaphors is here being violated and therefore this superficially created metaphor might cause confusion.
Table 2. Extended mapping of the number line metaphor

<table>
<thead>
<tr>
<th>Source domain (S’)</th>
<th>Target domain (M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standing somewhere on the line facing right or up[3]</td>
<td>default setting</td>
</tr>
<tr>
<td>Standing on the centre point on the line</td>
<td>zero</td>
</tr>
<tr>
<td>Standing on a point on the line relative to the origin</td>
<td>a number</td>
</tr>
<tr>
<td>Standing on a point on the line to the right or above the origin</td>
<td>a positive number</td>
</tr>
<tr>
<td>Standing on a point on the line to the left or below the origin</td>
<td>a negative number</td>
</tr>
<tr>
<td>A distance on the line</td>
<td>absolute value</td>
</tr>
<tr>
<td>Facing right or upwards and moving forward</td>
<td>adding a positive number</td>
</tr>
<tr>
<td>Facing right or upwards and moving backward</td>
<td>adding a negative number</td>
</tr>
<tr>
<td>Turning around and moving forward</td>
<td>subtracting a positive number</td>
</tr>
<tr>
<td>Turning around and moving backward</td>
<td>subtracting a negative number</td>
</tr>
</tbody>
</table>

Holding on to the thermometer as a metaphor (see table 3) entails a different extension. Adding a negative number is interpreted to be the same as subtracting a positive number; \( 3 + (-2) = 3 - 2 \), an interpretation which lacks meaning in the source domain. The mapping is no longer internally consistent since addition sometimes is seen as a motion along the line and sometimes need to be transformed into a subtraction. The subtraction \( (a - b) \) can be interpreted as the difference between \( a \) and \( b \), or the motion needed to get to \( a \) from \( b \). This is coherent with the mathematical definition of subtraction saying that \( (a - b) \) is the number \( x \) that solves the equation \( b + x = a \). The great difficulty here is that in our experience of the difference between two temperatures or two points on a line, we always conceive of that difference as an absolute value. The difference between the two numbers \(-4\) and \(+6\) is always spoken of as \(10\). If we are to understand that \( 6 - (-4) \) and \((-4) - 6 \) yield different answers we need to incorporate into the mapping the aspect of direction. \( 6 - (-4) = 10 \) because the temperature rises from \((-4)\) to \(6\), whereas \((-4) - 6 = (-10) \) because the temperature falls from \(6\) to \((-4)\). The direction here is from \(b\) to \(a\), which is from right to left in the expression. This is an ‘unnatural’ direction since we often intend the reverse interpretation: \(2 - 8\) usually reads ‘from two to eight’. Direction has been identified as one of the critical features for learning subtraction of negative numbers (Kullberg, 2006). It is, however, a feature closely connected to the use of certain metaphors and the interpretation of subtraction as difference. See table 3. This extension of the metaphor is inconsistent since for the subtraction \( a - b = c \), \( a \), \( b \) and \( c \) have different sources depending on whether they are positive or negative. If \( b \) is positive then \( a \) and \( c \) are referred to as temperatures and \( b \) is the change, but if \( b \) is negative then \( a \) and \( b \) are referred to as temperatures and \( c \) is the change.
Table 3. Extended mapping of the thermometer metaphor

<table>
<thead>
<tr>
<th>Source domain (S)</th>
<th>Target domain (M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centre point on the line (here called the origin)</td>
<td>zero</td>
</tr>
<tr>
<td>A point on the line relative to the origin</td>
<td>a number</td>
</tr>
<tr>
<td>A point on the line above the origin</td>
<td>a positive number</td>
</tr>
<tr>
<td>A point on the line below the origin</td>
<td>a negative number</td>
</tr>
<tr>
<td>A distance on the line</td>
<td>absolute value</td>
</tr>
<tr>
<td>Motion up (rising)</td>
<td>adding a positive number</td>
</tr>
<tr>
<td>Motion down (falling)</td>
<td>subtracting a positive number</td>
</tr>
<tr>
<td>A distance with direction between two points on the line</td>
<td>subtraction</td>
</tr>
</tbody>
</table>

To add a negative number a different metaphor is needed that will make it plausible that \( a + (-b) = a - b \) since there is no way a temperature can go up a negative number of degrees.

RESULTS

As we have seen, extending the metaphor may make the metaphor less functional for three different reasons:

- Loss of consistency within the metaphor itself
- Loss of coherence with connected metaphors
- Properties or structure being forced on the source domain that is not part of the experiential basis.

Expanding the metaphor “Arithmetic as Motion Along a Path” in either of the two ways described here entails difficulties. It remains to be shown whether what is gained by the extension compensates for what is lost in comprehension and applicability. Nevertheless it is essential that teachers are aware of the consequences of extensions like the ones described in this paper. There are of course alternative ways of extending the metaphor, and other models, referring to different grounding metaphors, that also need to be extended in order to cover up subtraction of negative numbers. A further analysis of such metaphors could be a helpful tool for teachers in their struggle of teaching this topic. Analyzing what happens to a grounding metaphor that is extended to include the operations multiplication and division with negative numbers is a different task again, but nonetheless important.

NOTES

1 The words representation and model are in this paper used in the same meaning.
2 When only dealing with positive numbers this is the starting point of the line.
3 If the calculation involves several operations it is necessary to return to default setting after every operation.
ADDITION AND SUBTRACTION OF NEGATIVE NUMBERS

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4. COMPLEXITY OF OPERATING BEYOND NAÏVE EMPIRICISM WHEN PROVING A CONJECTURED FORMULA FOR THE GENERAL TERM OF A GEOMETRICAL PATTERN

Three student teachers have collaborated on expressing the general term of a geometrical pattern. Their underlying logic of generalisation is naïve empiricism. They are then given the task of proving the conjectured formula. This paper deals with the complexity of understanding the disparity between formulas being conjectures based on empirical examples and formulas resulting from utilisation of structural relationships. The research question is: What is important for the development of an attitude of proof of generality in geometrical patterns?

THEORETICAL BACKGROUND

Guy Brousseau’s (1997) theory of didactical situations in mathematics is a scientific approach to the set of problems posed by the diffusion of mathematics, in which the specificity of the knowledge taught is engaged and plays a significant role. The theory is about an interplay between three instances; the teacher, the student (or rather, a group of students), and mathematics. It deals with how the teacher can organise situations which make it possible and meaningful for the student to develop some particular mathematical knowledge. Brousseau defines an adidactical situation to be a situation in which the student is enabled to use some knowledge to solve a problem without appealing to didactical reasoning [and] in the absence of any intentional direction [from the teacher] (p. 30). According to Brousseau, the devolution of an adidactical learning situation is the act by which the teacher makes the student accept the responsibility for an adidactical learning situation or for a problem, and the teacher accepts the consequences of the transfer of this responsibility (p. 230). The student cannot solve any adidactical situation immediately; the teacher arranges an adidactical situation which the student can handle, which means that it is based on the student’s prior knowledge and experiences.

There are four different types of didactical situations, depending on the phases of the ‘game’ that the teacher plays with the student-milieu system; situations of action, of formulation, of validation, and of institutionalisation (Brousseau, 1997, pp. 161–162). Each situation depends on the accomplishment of the antecedent situation. The episode described in this paper is in the phase of validation.

The situation of validation is about establishment of theorems. For a model or a theory to be qualified as a valid strategy of winning the ‘game’ there is a need.
for argumentation. Brousseau (1997) claims that to state a theorem is not to communicate information. It is always to confirm that what one says is true in a certain system; to be ready to prove that it is true. He emphasises that it is therefore not a question only of the student’s ‘knowing’ mathematics, but of using it as a reason for accepting or rejecting a proposition (a theorem), a strategy, a model, (...) which requires an attitude of proof (p. 15). Brousseau asserts that this attitude of proof is not innate and therefore needs to be developed and sustained through specific didactical situations; situations of validation.

In mathematics, the ‘why’ cannot be learned only by reference to the authority of the adult. Truth cannot be conformity to the rule, to social convention like the ‘beautiful’ and the ‘good’. It requires an adherence, a personal conviction, an internalization which by definition cannot be received from others without losing its very value. (Brousseau, 1997, p. 15)

A goal in generalising geometric and numeric patterns is to obtain a new result. Luis Radford (1996) claims that conceived in this form, generalisation is not a concept, but a procedure allowing for generation of a new result (a conclusion), based on observed facts. He highlights that one of the most significant characteristics of generalisation is its logical nature which makes the conclusion possible. This means that the process of generalisation is closely connected to that of justification and proof. The underlying logic of generalisation can be of various types, depending on the student’s mathematical thinking.

Nicolas Balacheff (1988) has identified four types of proof in pupils’ practice of school mathematics: Naïve empiricism, the crucial experiment, the generic example, and the thought experiment. Many students think that some examples are sufficient to justify the conclusion, a stance categorised by Balacheff as naïve empiricism. Others may think that the validity of a conclusion is accomplished by testing it with a special term of the sequence, for instance the 100th term, a stance categorised as a crucial experiment. The generic example involves making explicit the reasons for the truth of an assertion by means of operations on an object that is a representative of the class of elements considered. A generic example is an example of something; the validity of a hypothesis is argued for by the characteristic properties of this example. The thought experiment is, according to Balacheff, a conceptual proof which requires that the one who produces the proof is giving up the actual object for the class of objects on which relations and operations are to be described. In order for this to happen, language must become a tool for logical deductions and not just a means of communication (Balacheff, p. 217).

Balacheff (1988) emphasises that a proof by naïve empiricism or by a crucial experiment does not establish the truth of an assertion; he talks of ‘proof’ only because they are recognised as such by their producers. The generic example and the thought experiment are, however, valid proofs. It involves a fundamental shift in the students’ reasoning underlying these proofs: It is no longer a matter of ‘showing’ the result is true because ‘it works’; rather it concerns establishing the necessary nature of its truth by giving reasons (Balacheff, p. 218).
COMPLEXITY OF OPERATING BEYOND NAÏVE EMPIRICISM

METHODS

The participants in the research reported are three female student teachers, Sofie, Ida, and Alise (pseudonyms), and a teacher educator [1] in mathematics. At the time of the data collection the students were in the second semester the first academic year of the teacher education programme and had been collaborating on several assignments in different topics. The course, in which the students’ work was observed, is a compulsory course in mathematics education (30 ECTS credits). The episode described is a video-recorded small-group work session at the university college, in which the students work collaboratively on a task designed by their mathematics teacher. The students are placed in a group room adjacent to a big classroom, in which the rest of the class is working on the same task.

ANALYSIS

The students have been working for 50 minutes on a generalisation task based on a geometrical pattern of which the first three figures were given (Figure 1). The aim of the task was to express, as a mathematical statement, what the figures seem to show.

![Figure 1. The first figures in the pattern that has been generalised by the students](image)

With some help from the teacher, the students have formulated that the sum of the $n$ first odd numbers equals the $n$-th square number. Their mathematical statement is represented in algebraic symbols by the equality $1 + 3 + 5 + \cdots + 2n - 1 = n^2$ (Formula 1) and has been verified for $n \in \{2, 3, 4, 5\}$. The teacher asserts that because the relationship expressed by this formula is not obvious or a matter of course, we need to prove mathematically that it is true in all cases. He continues:

For the time being we have a hypothesis because we have tried with three and four and five and perhaps more, but we still have tried only for a finite number of cases. We just have some examples, so at present it is about raising ourselves to a more superior plane and see the structure. (Teacher, Turn 392)

This utterance by the teacher marks a transition to the next part of the task (Figure 2), in which the teacher has delineated a process of proving Formula 1. When the students start to work on the task, Sofie makes a drawing (Figure 3) and records the number of black and red [2] quadrates in each rectangle, as shown in Table 1. On the basis of the numerical facts (addends in parentheses) in this table, using trial and error, the students come up with an explicit formula that represents the total number of quadrates in the $n$-th rectangle, $n^2 + n \cdot (n - 1)$ (Formula 2).
The students are seen to do what the task asks for; establishing an explicit formula for the total number of quadrates in the \( n \)-th rectangle. However, the way they establish the formula, generalising by trial and error from numerical facts, indicates that they do not comprehend how the task is designed to play a part in proving the equality 
\[
1 + 3 + 5 + \cdots + 2n - 1 = n^2. 
\]

The mathematical statement that you have attained above is only based on a few observations. It is therefore just a conjecture. In order for it to become a justified statement, it needs to be proven. We will now look at a possible way of doing this.

e) Build or draw (using two colours or symbols) the next figure in this pattern.
f) Find an explicit formula for the total number of quadrates in the \( n \)-th figure. Then find an explicit formula for the number of \( \square \) in the \( n \)-th figure.
g) Use this to find the number of \( \blacksquare \) in the \( n \)-th figure.
h) Discuss if the conjecture from earlier in the task can be considered proven now.

\begin{figure}
\centering
\includegraphics[width=0.3\textwidth]{image}
\caption{Part 2 of task given to the students}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.3\textwidth]{image}
\caption{Sofie’s illustration of the first four rectangles in the pattern}
\end{figure}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{number} & \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \textbf{5} \\
\hline
\textbf{1} & \textbf{(4 + 2)} & \textbf{(9 + 6)} & \textbf{(16 + 12)} & \textbf{2} & \textbf{3} & \textbf{4} & \textbf{5} \\
\hline
\end{tabular}
\caption{Sofie’s table showing the number of black and red quadrates in the rectangles}
\end{table}

The point, as delineated in the task, is to calculate the area of the \( n \)-th rectangle to find its total number of quadrates, and to use the formula for the \( n \)-th triangle number to find the number of white quadrates in the \( n \)-th rectangle. Because of the structure of the \( n \)-th rectangle, the number of black squares is observed to be the difference between the total number of quadrates, \( n(2n - 1) \), and the number of white quadrates, \( n(n - 1) \). Simplifying the algebraic expression for the difference yield \( n(2n - 1) - n(n - 1) = n^2 \). This shows that the total number of black quadrates in the \( n \)-th rectangle, represented by the sum of the first \( n \) odd numbers, equals \( n \) squared, which fulfils the proof of Formula 1.
The design of the task presupposes that the students recognise (in the rectangles) the structure of triangle numbers and that the formula for the $n$-th triangle number is prior knowledge. There is no evidence that the students recognise the structure of the white quadrates to be triangle numbers, which may explain that they generalise the number of quadrates in the rectangles on the basis of numerical facts, not on the basis of structure. Formula 2 is by the students established on the basis of assuming that the number of black quadrates (sum of odd numbers) equals $n^2$. Because this is actually what they are supposed to prove, the process of proving Formula 1 collapses.

The transcript [3] below shows how Formula 2 is justified through empirical examples.

**Justification of generality by naïve empiricism and a crucial experiment**

468 Ida: Four plus two?

469 Sofie: Yes, if you look here at the second (figure) (Ida: uh huh) then we have got two three four black (Ida: yes) and two red (Sofie counts black and red quadrates in the second figure in her pattern as drawn in Figure 1 above) (Ida: uh huh). Hence I write four plus two.

470 Ida: Ok.

471 Sofie: Figure three, here you have got one two three four.. five six seven eight nine.. black

472 Ida: plus six

473 Sofie: one two three four five six… (pause 9 sec.). And here you have got one two three four five six yes seven… no nine ten eleven twelve thirteen fourteen fifteen sixteen plus twelve (Ida: uh huh). Sixteen black plus twelve red… (Pause 20 sec.)

474 Sofie: We can indeed try to make or calculate with seven and afterwards draw and look if it is correct. (Ida: uh huh).

(Three turns)

478 Ida: Seven times six is 42. 49 plus 42. (Sofie draws a tower with $1+3+5+7+9+11+13$ black quadrates and then picks up the red pen). No, just continue to draw. (Ida counts how many rows the tower spans).

479 Sofie: What?

480 Ida: I will just check how many there are. (She counts the number of quadrates at the bottom row). Thirteen.

481 Alise: It is seven times thirteen.

482 Ida: Yes.
Alise: Haven’t calculated this (product). (Sofie colours with red the rest of the rectangle. Ida verifies that \(49 + 42\) equals \(7 \cdot 13\) by calculation).

Ida: It becomes true. This is correct. Good, Sofie! (excited voice).

Sofie: Ok then… (smiles)

Ida: Was it correct for more values? Have you done several? (to Alise)

Alise: No, I have only checked for (figure number) four. There it is correct.

Ida: Yes, and it’s correct for seven

(Draws a frame around the formula in her notes like this:

\[
\begin{align*}
\quad & n^2 + n \cdot (n - 1) \\
\end{align*}
\]

Turns 469, 471, and 473 show that Sofie is concerned about the number of black and red quadrates; she is not focusing on the structure of the figures. This is signified by her counting the black and the red quadrates from the first one in the second, third, and fourth figure in the pattern. She verifies that the addends \(n^2\) and \(n(n - 1)\) in Formula 2 correspond to the numbers of black and red quadrates in the rectangles, respectively. Justification of Formula 2 by Sofie in turns 468 – 473 is seen to be done by naïve empiricism for \(n \in \{2, 3, 4\}\). In turn 474 Sofie suggests that a crucial experiment is carried out for \(n = 7\). Ida’s verification of the fact that the sum of the addends in the formula when \(n = 7\) equals the product of the breath and length of the seventh rectangle (as commented in turn 483) induces her to assume that the conjectured formula is correct. The conclusiveness of Ida’s assumption is indicated by her exited voice in turn 484 and her drawing of a frame around the formula, as commented in turn 488.

Intricacy of explaining what a valid proof involves: The thought experiment

After a break the teacher, on his own initiative, joins the students. The students let him know that they are about to find a formula for the number of black quadrates in the \(n\)-th rectangle. When the teacher asks what Formula 2 in Ida’s notes is about, they answer that it represents the total number of quadrates in the \(n\)-th rectangle. They reveal that they have verified the formula for \(n \in \{2, 3, 4, 7\}\) and concluded that it therefore must be correct for all \(n\). This utterance is succeeded by the teacher’s claim that a number of examples (regardless how many) can never prove that a property is true in all cases. The following exchange takes place:
In order to be quite sure, we have to raise ourselves in the sense of seeing a superior structure in our system. What we now believe is that the number of dark quadrates is $n$ squared. But we don’t know it because we have only verified it in some particular cases. (Sofie looks at the teacher with slightly lifted eyebrows, Alise leans the head in her left arm).

So this isn’t quite correct? (Laughs and points at Formula 2)

Not correct in the sense that you can kind of defend it in a trial yet.

It is not the one that we were supposed to obtain.

But it is correct… but what?

Correct, in what way is it correct?

It is correct if we put the numbers in (the formula). Put them in.

Yes, but then you have (verified) it only for some examples.

Yes, but this applies for all the formulas that we invent, doesn’t it?

Well, so this is really the first time we have challenged you to go beyond… No we have in fact done that before. But there is actually a leap from testing some examples and getting a conjecture, to really proving that it is true. And in order to prove that it really is true, we have to see a structure that is independent of choosing three or five or seven, which is more about the whole (Ida nods) and the structural relationship in a way. And that is not so obvious to see in these figures (points at Figure 1), but the reason why I have added the white quadrates here was to help you… it may be easier to identify the structure of this pattern (in Figure 2) than of that pattern (Figure 1).

I can’t really get the picture here now. I don’t quite understand… What… what are we supposed to get at then?

Turn 549, which basically repeats the teacher’s utterance in turn 392, is the teacher’s reaction to the students’ empirical reasoning. He is here seen to argue for a thought experiment to be accomplished. An attempt to explain what he means by “raising ourselves to a more superior plane and see the structure” (392) and “seeing a superior structure in our system” (549) is made when he says that “in order to prove that it really is true, we have to see a structure that is independent of
choosing three or five or seven, which is more about the whole and the structural relationship in a way (turn 558).

The teacher’s focus is on proving Formula 1; how to use the structure of a rectangle and prior knowledge of triangle numbers to establish a connection between sums of odd numbers and square numbers. The students’ focus, however, is on Formula 2, of whose validity they are convinced by empirical reasons. Sofie’s and Alise’s gestures as described in turn 549 indicate that they find the teacher’s utterance (549) surprising or incomprehensible. Ida’s utterances in turns 550 and 552 signify that she interprets the teacher to mean that their obtained formula is wrong (on the symbolisation level). Sofie’s utterances (turns 553, 557, and 559) show that she does not understand what the teacher means and what they are supposed to be after, if it is not their algebraic expression (Formula 2).

When the validity of a formula is of concern, the way the formula is established is significant. There is a disparity between a formula being a conjecture based on empirical examples and a formula resulting from using structural relationships of these examples. This disparity does not need to be unique at the symbolisation level, a fact that contributes to the complexity of understanding that it is not the formula as such that may be inadequate – the point is how it is derived. Formula 2 does indeed represent the total number of quadrates in the general term of the sequence of rectangles, but it is derived by trial and error, not by reasoning on the basis of structural relationships.

The teacher uses the notions structure and structural relationship in his explication of what a valid proof involves, but these notions are not further explained until turns (577 – 597). Until this elaboration, the students are interpreted not to have a clear conception of what ‘structure’ means and how it can be used to reason beyond naïve empiricism. The inadequacy of Formula 2 is by Sofie not understood during the work session. This is evidenced by her asking: “What was wrong with the first one (about Formula 2) that we made? That it wasn’t.. correct?” (turn 704).

**DISCUSSION**

In the analysed episode the students did not know what the teacher meant by ‘utilising structural relationships’. Consequently, they were unable to conceive of the disparity between a formula being a conjecture based on empirical examples and a formula resulting from using structural relationships. The students were seen to show that their formula was correct because ‘it worked’, rather than establishing the necessary nature of its truth by giving reasons. The teacher’s formulations, “raising ourselves to a more superior plane and see the structure” and “raise ourselves in the sense of seeing a superior structure”, were attempts to describe the fundamental shift from empirical to theoretical reasoning.

The teacher has devolved to the students an adidactical situation that presupposes prior knowledge about triangle numbers. There is however a problem caused by the students not recognising the triangle numbers in the given sequence of figures, which means that the students cannot handle the adidactical situation. The situation in which the students engage is a situation of validation. But there is a problem
caused by the students’ not being able to use knowledge about structural relationships as a reason for accepting the conjectured formula.

The analysis points at the necessity of establishing the meaning of the notions ‘structure’ and ‘relationships’ on objects. This is important for the development of an attitude of (valid) proof and contributes to answering the research question posed in the paper.

NOTES

1 The student teachers will be referred to as ‘students’, and the teacher educator as the ‘teacher’.
2 Sofie colours with red the quadrates that are white in the given pattern.
3 Codes used for transcription
   … Pause (up to 3 seconds)
   … A small hesitation
   *Italics* Emphasis

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