MATHEMATISATION AND DEMATHEMATISATION
Mathematisation and Demathematisation

Social, Philosophical and Educational Ramifications

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DEDICATION

*Festschrift* is a German word used to denote a volume of writings by various authors which is presented as a tribute, nowadays usually to a prominent scholar.

The first Festschrift was published in 1640 by Gregor Ritzsch in Leipzig on the occasion of the bicentenary of the invention of the printing press ["Jubilaeum Typographorum oder zweyhundertjähriges Buchdrucker-Jubelfest"].

Poets from all over Germany contributed.

The presentation of such a Festschrift has become a tradition in German-speaking academia and is made by fellow scholars as a token of their affection and respect, and is considered a great honour.

We are pleased to be able to continue this tradition, although in this case, as the title might suggest, the contributions can hardly be considered poetic.

This book is dedicated to a unique personality, and is the occasion of our being able to express our warm gratitude for all she has done for us: her ability to fire our imaginations, and through stimulating discussions show us that, without a transdisciplinary approach, the crucial issues cannot be understood, and that research can only be relevant if it is sustained and nourished by a vision. She is a scholar who did not allow us to forget that mathematics education should not be separated from its political and social context. Her introducing us to the international scientific community saved us from the fate of becoming too deeply rooted in the German community of mathematics educators.

This book is dedicated to Professor Christine Keitel on the occasion of her birthday.

Uwe Gellert and Eva Jablonka  (January, 2007)
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MATHEMATISATION – DEMATHEMATISATION

This introductory chapter attempts to discuss and thereby clarify the meanings of mathematisation and demathematisation. Given the sufficiently cryptic character of the title of this volume, such a clarification seems particularly important.

The title of this volume is motivated by our belief that the notions of mathematisation and demathematisation describe two phenomena, which are dialectically related. In our view, the power of the concept of mathematisation in terms of its heuristic value and theoretical fruitfulness might be considerably increased if the reverse process – demathematisation – were conceptually integrated.

Scholars from the diverse strands of research have made use of the notion of mathematisation in order to coin a process in which something is being rendered more mathematical than it has been before. There is a tradition of discussing mathematisation as a didactic principle, for example in the context of Realistic Mathematics Education in the Netherlands. On the other hand, some use the concept of mathematisation for the description and analysis of the social, economical or political processes in which relationships between the participants become increasingly formal. Because these apparently different conceptual understandings of mathematisation are used in parallel within the field of mathematics education, it seems appropriate to first point out conceptual differences.

A common concern of all authors of this volume is the issue of social availability of mathematical knowledge. Reference is made more or less explicitly to the educational dimension of processes of mathematisation and demathematisation. Together, the contributions reveal a rather complex picture: They draw attention to the importance of clarifying epistemological, societal and ideological issues as a prerequisite for a discussion of curriculum. Such a view implies a critique of those curricular conceptions focussing on mathematisation, which lack adequate forethought. In line with this view, in this chapter, after elucidating the concepts of “horizontal and vertical mathematisation” and “mathematisation in the course of mathematical modelling”, we discuss in more detail the work of scholars who concentrate on analysing mathematisation and demathematisation as a social process.
MATHEMATISATION AS A DIDACTIC PRINCIPLE

It is in fact a very old tradition to use a simple description of an everyday or professional activity as a paradigmatic representation of a class of similar problems for the purpose of introducing a mathematical method for solving them. These mathematical tasks contain no redundant information, no data is missing, the answer is well defined and the results are never brought into operation. The term mathematisation is used by some to describe the activity of the students who are dealing with these types of word problems. The object, which is to be mathematised, need not be as simple as a traditional textbook problem, but can also consist of a description of a more or less authentic out-of-school situation, which by its alleged authenticity is obviously more complex than a simple textbook problem.

**Horizontal and vertical mathematisation**

Mathematics lessons often consist of students being introduced and accustomed to read and solve textbook problems, many of which are contextualised. One of the reasons for the difficulties teachers and students, who are faced with “realistic” problems, experience, is that school mathematics, if about optimisation when shopping, for example, is neither mathematics nor shopping.

Dowling (1996, 1998) describes the effects of this didactic practice as *the myth of participation* and *the myth of reference*. He identifies two distinct messages in school mathematics texts. One, the “myth of participation”, suggests that mathematics deals with the public domain. Realistic problem situations in the form of narratives from the perspective of a person acting in a practical situation are used as an introduction into the esoteric domain of mathematics. This strategy supports the view that everyday practical knowledge is constitutive of mathematical knowledge. But it is likely that this knowledge turns out to be an insufficient base for solving the contextualised problems because the mathematical solution has a structure that is different from any practical solution. The second message is the “myth of reference”. It is conveyed through problem settings that are constructed mathematically and only retain a trace of non-mathematical significations. It does not remain possible for the learner to evaluate the solution of the problems from a practical point of view. However, the ‘recipients of this message’ are expected to believe that behaving mathematically in such situations would be in their own interests. According to Dowling, both of these messages are myths: The application of mathematics in practical (domestic) activities conceals its generalisability, and the expression of domestic activities in mathematical terms conceals the cultural arbitrariness of the mathematical principles projected into them.

In both cases, students need to transcend the everyday context, in which a problem is formulated, by converting the semantic structure of the text into a mathematical description. Of course, since problems given in textbooks generally
do not claim to mirror problematic situations authentically, the mathematisation required from the students is essentially an artificial activity.

According to Treffers (1987) the process of formulating a mathematical description might be called a horizontal mathematisation. The semantic transfer occurs horizontally between different domains of description: from the public to the esoteric. In the mathematics classroom, particularly in the lower grades, it is typical to switch between these two domains. First mathematical concepts, like the cardinal and ordinal aspect of numbers, are built from activities or relationships within the public sphere. These concepts, as Freudenthal (1983) points out, help to organise many everyday phenomena. Also in the higher grades, mathematical meaning is often developed from out-of-school experience.

Adler (2001) provides telling empirical accounts of secondary mathematics teachers being confronted with this changing between the public and esoteric language of description. A dilemma emerges: If classroom talk concentrates on the public language of description in order to help students construct meaning, then the mathematical knowledge of students tends to remain in the public domain of its origin. If, on the other hand, the classroom talk is mainly esoteric, then the individual construction of meaning appears to be more difficult. A similar point is made by Vithal and Skovsmose (1997) in respect to ethnomathematics. They argue that career aspirations of students and parents would be ignored if mathematics education concentrated too long on local knowledge rather than providing the opportunity for students to enter the esoteric domain of academic mathematics.

However, it can be argued theoretically that the switching itself is intrinsically problematic, because everyday knowledge is often contradictory across contexts. According to Bernstein (1996, 1999), everyday knowledge, or common-sense knowledge, is context-dependent, segmentally organised and consistent within each segment, but segments overlap and knowledge organisations often do not match. Bernstein calls the discourse referring to this type of knowledge a horizontal discourse, compared to a vertical discourse, which takes the form of a systematically principled and hierarchically organised structure, such as in the sciences, or of a specialised language, as in the social sciences and humanities. Horizontal mathematisation as used by Treffers (1987) can be described, within Bernstein’s theoretical framework, as a transfer between horizontal and vertical discourse. It is problematic that horizontal knowledge remains contradictory across its segments, whereas vertical knowledge strives for coherence and a principled structure. In the process of horizontal mathematisation, the contradictoriness of the horizontal discourse is mostly ignored. The fiction is, that abstraction from extra-mathematical contexts to mathematical concepts and structures is possible and straightforward, but, actually, this process is a step from the contradictory world to a coherently organised esoteric sphere that has long since cut off its everyday roots.

Once the classroom activities occur within the esoteric domain of academic mathematics, e.g. defining concepts and proofing theorems, students get introduced into vertical discourse. This is what Treffers (1987) calls a vertical mathematisation. It is essentially an introduction into the organised structure of
mathematical knowledge. Thus, Treffers’s separation of vertical and horizontal mathematisation is reminiscent of Bernstein’s discourses.

Horizontal and vertical mathematisation – in the curricular conception laid out by Treffers (1987, 1991) and colleagues (de Lange, 1996; Gravemeijer, 1994; Streefland, 1991; Treffers & Goffree, 1985) – define mathematisation as a multi-step activity for students, aiming at the exploration of mathematical structures. The focus is on the mathematics, not on the ‘realistic’ situations from which the mathematisation is hoped to be derived. Everyday situations are valued mostly for the alleged motivation and illustration that they invoke. In short, this kind of mathematisation is concerned with its ends (which reflect the curriculum) rather than with the process itself.

Mathematisation as a didactic principle is often underpinned by the assumption that the historical development of basic mathematical concepts can be used as a guideline for the learning process of an individual:

It is a characteristic of mathematical activity, according to the realistic view, that the build up of elementary skills can take place via a process of reinvention or independent construction. (Treffers, 1991: 31)

The “realistic view” is a rudiment of the genetic principle in epistemology as formulated by developmental psychology. This principle postulates parallelism of ontogenesis and historiogenesis. Damerow, in this volume, develops a framework for a historical epistemology of the concept of number in order to answer the question of whether in the history of mathematical knowledge stages of development can be identified which at the same time constitute and constrain individual cognition. Arithmetical thought is reconstructed as an outcome of the material culture of calculation.

From ‘real world models’ to mathematical models

The process of translation from ‘reality’ to mathematics and back is in the focus of curricular conceptions figuring under the title mathematical modelling (de Lange, Keitel, Huntley, & Niss, 1993; Houston, Blum, Huntley, & Neill, 1997; Matos, Blum, Houston, & Carreira, 2001). In the mathematical modelling perspective, the teaching and learning of mathematics is organised along a didactically simplified view of applied mathematics. The fundamental idea of this approach is that students should be introduced to, and involved in, what are seen as the core subprocesses that applied research runs through when tackling authentic social or technical problems.

In order to give a structured description of ‘real world problem solving’, the ‘world’ is separated into two spheres: ‘reality’ and ‘mathematics’. A ‘real world problem’, of course, is located in a ‘real world situation’. According to the modelling perspective, the first step in coming to terms with the ‘real world problem’ consists of a problem description based on everyday knowledge of the situation. This description, in theory, is non-mathematical. It is regarded as a ‘real world model’ of a ‘real world situation’ in which the criteria for acceptable
solutions are made explicit. Take as an example the ‘real world situation’ of some film enthusiasts discussing the scariest movie ever seen (‘real world problem’). After a lengthy debate they might agree on a set of criteria, which a movie should meet in order to be scary: Suspense is important, a good dose of realism as well, an emerging feeling of subjection, and some level of gore are all supposed to produce the best effects. These criteria make for the non-mathematical ‘real world model’.

When the ‘real world model’ is converted into a formal description, e.g. a mathematical equation, the sphere of ‘reality’ has been left, and a ‘mathematical model’ is created. The mathematical description of the ‘real world model’ is the result of a translation from ‘reality’ to ‘mathematics’. This ‘translation’ into the ‘mathematical model’ is by no means straightforward. It often includes, firstly, quantifying different non-mathematical characteristics and, secondly, relating these to each other mathematically. In the example above, it is not clear from the outset, how a good dose of realism and some level of gore might be best expressed mathematically, and whether they are equally important. Within the curricular conception of mathematical modelling, the process of ‘translation’ from the ‘real world model’ to the mathematical model is called mathematisation.

The formal problem description allows for mathematical concepts and algorithms to produce mathematical results. According to this simplified description of modelling, these results need to be tested against the ‘real world situation’ (or its ‘real world model’) from which the complex process started. Since the results often do not match the problems, reformulations of the ‘real world model’ and modified mathematisations might be necessary. As a consequence mathematical modelling is described as a circular process. Since within a classroom activity the results are never put into operation there is no real problem of validation.

It has been argued elsewhere (Gellert, Jablonka, & Keitel, 2001), that the epistemological underpinnings of applied mathematics are not adequately represented in the conception of mathematical modelling as described above. The symbolic technology at hand, as well as measuring devices, and domain-specific constraints and theories, all influence the form a ‘real world model’ will take. So the ‘real world model’ is by no means a description of the ‘real world problem’ (Cooper & Dunne, 2000; Gates & Vistro-Yu, 2003; Jablonka, 1997).

In the mathematics classroom, the fiction that ‘real world models’ are independent of the mathematical technology at hand is even harder to maintain: How can students construct non-mathematical ‘real world models’ according to their interests in distinct aspects of the situation and at the same time be aware of the fact that they are sitting in a mathematics lesson, and that they will have to mathematise and solve the ‘real world models’ they are building by means of their (limited) mathematical knowledge? The models students construct tend to relate to the mathematical tools they have developed shortly beforehand.

Mathematisation within the circular process of mathematical modelling is – epistemologically regarded – a potentially misleading construct and it is – pedagogically – of debatable value. On the one hand, the circular model of mathematical modelling adequately acknowledges the contingencies of problem
definition and formalisation; on the other hand, it tends to obscure the informative power of mathematics. Mathematics is not only the sphere where formalised problems find their solutions; mathematics is from the outset the vantage point from which the problems are construed.

MATHEMATISATION AS A SOCIAL PROCESS

Mathematisation, in the context of mathematical modelling as a didactic principle, falls short of grasping the fact that the ‘world’ students live in is already interspersed with constructions and processes based on mathematics. Mathematics is a means for the generation of new realities not only by providing descriptions of ‘real world situations’, but also by colonising, permeating and transforming reality. Models become the reality, which they set out to model. Consequently, any discussion of mathematisation has to take into account the social process by which mathematical models are developed, implemented, accepted, and obscured.

Mathematics shaping society

According to Davis and Hersh (1986), we are living in a Descartesian world. Mathematics has penetrated many if not most parts of our lives. It has capitalised on its abstract consideration of number, space, time, pattern, structure and its deductive course of argument, thus gaining an enormous descriptive, predictive and prescriptive power. Within science, there is hardly any theory, which is not formulated in mathematical terms. In sociology, psychology and education, quantitative studies are highly valued. It is hardly impossible to understand any theory of economics without a solid mathematical background. In all these fields of human activity mathematics can be regarded as the grammar of the particular scientific discourse and as a universal tool. However, mathematics being the grammar or the tool implies that the characteristics of this grammar strongly influence the development of the fields in which the use of mathematics is made. It turns out to be difficult, if not impossible, to integrate any idea that cannot be formulated in mathematical terms into an accepted body of mathematically formulated theories.

The impact of mathematics is by no means restricted to scientific activity. Mathematics-based decisions affect the social interactions in technological societies on many levels (for a long list of examples see Davis & Hersh, 1986: 120-121). On the level of the national distribution of state salaries, pensions, and social benefits, political decisions often are made and communicated on the basis of formulae and diagrams, which themselves rely on mathematical extrapolations of demographical and economic data provided by experts. On the level of interpersonal relations, mathematics-based communication technologies have already changed the habits and styles of private conversations. Of course, the mathematics often is invisible, as in mobile phones and Internet chat forums, or it is just recognised on the surface as a medium of presentation.
By drawing attention to the many ways in which mathematics shapes society and exerts considerable influence on our everyday lives, Davis and Hersh (1986) introduce the concept of mathematisation, denoting both the process and the product of ongoing formalisation. Davis (1989) focuses on another crucial aspect of mathematisation that has not yet received much attention: Is mathematisation a naturally occurring phenomenon or, if not, who is in charge, and whose intentions become realised?

Living with realised abstraction

Mathematics, when creating and using abstract concepts, presupposes abstraction. The basic mathematical abstractions (particularly such as the number system) are not only abstractions in the mind, but also in action. As Sohn-Rethel (1978) argues, the abstraction of the exchange value of goods is performed in trading situations, as is the abstraction of a general quantified measure for goods which is independent from their special qualities. He calls an abstraction, that is not thought but articulated in a social action, a real abstraction. Mathematics, as Freudenthal (1983) points to, has developed over the centuries by reflecting on these real abstractions – a process that could be reversed for educational purposes in form of a didactical phenomenology. Such a reflection on real abstractions happens in the mind as a thinking abstraction (Sohn-Rethel, 1978). Since thinking abstractions do not take place in the action of people, but in their imagination, they are extremely flexible. Mathematical thinking has the power of hypothetical reasoning: It is possible to calculate some consequences of actions before these are carried out. Nobody needs to be afraid of the immediate consequences of any thinking abstraction.

In the long run, however, the world of thinking abstractions transforms back to what Keitel, Kotzmann and Skovsmose (1993) describe as a system of implicit knowledge. In many cases, we are neither aware of the circumstances under which a particular thinking abstraction has been processed, nor of the purposes for their initiation. The social origins and the history of many of the mathematisations Davis and Hersh (1986) describe are immersed. Technology, including social technology, functions as a black box – and the constitutive abstraction needs not to be reflected upon anymore. The substitution of abstraction processes by black boxes produces what Keitel, Kotzmann and Skovsmose (1993) call implicit mathematics. In order to stress the point that thinking abstractions – after having been derived from real abstractions and developed by hypothetical trial and error – shape the technology with the help of which we organise much of our life, Keitel, Kotzmann and Skovsmose (1993) introduce the notion of realised abstraction. Thinking abstractions become materialised, they become part of our reality, and most of the time we do not ask where they come from or what they are – there is no necessity for doing so. Our time-space-money-system is a striking example for the implicitness of the underlying abstraction processes. We are dealing with our system of space, time and money as if no other ways were possible. Indeed, these coordinates of our life seem natural to us. Where people share other conceptions
about these (e.g., Harris, 1991) – while living in the same ‘global village’ – their constructions appear virtually incomprehensible to us.

**PROCESS AND EFFECTS OF MATHEMATISATION/ DEMATHEMATISATION**

The concept of realised abstraction may make us understand that the mathematisation of our world is just one side of the coin. On the other side, most people need to carry out less mathematics explicitly. Technology is more effective. The existence of materialised mathematics in the form of black boxes reduces the importance of mathematical skills and knowledge for the individual’s professional and social life. A demathematisation process is taking place (Keitel, 1989; Keitel, Kotzmann, & Skovsmose, 1993):

This term [demathematisation] also refers to the trivialisation and devaluation which accompany the development of materialized mathematics: mathematical skills and knowledge acquired in schools and which in former time served as a prerequisite of vocation and daily life lose their importance, and become superfluous as machines better execute most of these mathematical operations. (Keitel, Kotzmann, & Skovsmose, 1993: 251)

The process of demathematisation affects strongly the values associated with different kinds of knowledge and skills. For the user of technology it becomes more important to, first of all, simply trust the black box and, then, to know when and how to use it – for whatever purpose. If it turns out to be inefficient, ineffective, erroneous or disastrous, nobody can be blamed – it was the technology’s fault. From this perspective, demathematisation reduces the feeling for, and the acceptance of, responsibility: a car’s antilock braking system is taken as a licence for driving fast; formal assessment of personal creditworthiness ensures that a loan is not approved to the wrong people. When faults still occur, then this is a request for a better technology – technology that is designed to be foolproof.

Curiously, demathematisation is also connected with the myth of the infallibility of technology. When airplanes crash or nuclear power stations run into problems, it is often attributed to human error. However, safety precautions and regulations of such risk technologies are the product of balancing the probability and impact of accidents against the costs of preventive measures. Of course, this is again a matter of realised abstraction.

Chevallard (1989; reprinted in this volume) draws attention to the importance of a process he describes as follows:

Implicit mathematics are formerly explicit mathematics that have become “embodied”, “crystallized” or “frozen” in objects of all kinds – mathematical and non-mathematical, material and non-material –, for the production of which they have been used and “consumed”. (Chevallard, 1989: 50)
The theory of how sophisticated mathematical techniques, that were developed by “mathematical workers”, become simplified by algorithmisation and crystallized in technological tools, is reminiscent of Karl Marx’s observations on machinery:

The machine proper is therefore a mechanism that, after being set in motion, performs with its tools the same operations that were formerly done by the workman with similar tools. Whether the motive power is derived from man, or from some other machine, makes no difference in this respect. From the moment that the tool proper is taken from man, and fitted into a mechanism, a machine takes the place of a mere implement. The difference strikes one at once, even in those cases where man himself continues to be the prime mover. (Marx, 1961: 374).

Chevallard (1989; reprinted in this volume) describes the dialectic between implicit and explicit mathematics:

The greatest achievement of mathematics, one which is immediately geared to their intrinsic progress, can paradoxically be seen in the never-ending, two-fold process of (explicit) demathematising of social practices and (implicit) mathematising of socially produced objects and techniques. (Chevallard, 1989: 52)

It is indeed this “paradox”, which has to be the starting point of any discussion of the value of mathematical skills for an individual (e.g. Coben, 2003: 47-53; FitzSimons, 2002a, 2002b; Jablonka, 2003). As issues of curriculum became increasingly important with the development of mathematical technology, some scholars from the field of mathematics education have continued to explore the relationship between mathematisation and demathematisation. This ongoing discussion includes issues of the relationship between mathematics, society and technology; of a shift of values linked to mathematical knowledge; and of the importance of a democratic competence within demathematised societies.

The following discussion does not intend to be a comprehensive literature review, but to provide a base for further exploration by concentrating on some essential work focussing on those issues.

Mathematisation/ demathematisation through technology

Drawing on Mumford (1967), Keitel (1989) illustrates the role and possible effects of technology by the example of the mechanical clock. The construction of the clock is based on the perception of the movement of the planetary system:

This approach is generalized and condensed to a mathematical model, transformed into a technological structure, and as such installed outside its original limited realm of significance. Earlier human perceptions of time, which had grown out of both individual and collective experiences and remained bound and restricted to these, were now rivalled and ultimately substituted for by the novel way of perceiving time. (Keitel, 1989: 9)
The first effect of this technology is a mathematisation that makes it possible to measure time precisely and independently from the quality of the processes measured. The abstraction of comparability is presupposed. The meaning of time passing slowly in some instances, and running more quickly in others, can then only be understood metaphorically. The objective character of the mechanical clock denies subjective experience of time. The specific (subjective) situation, in which time is measured, has lost its relevance. A formalisation has taken place. Time is no longer valid as a concrete sensory experience (cf. Keitel, 1989).

This objectification and formalisation still has tremendous implications: Time is regarded as the sum of arbitrarily regular units. The idea of the linearity of time is associated with the concept of progress and endless evolution. Mathematics as the grammar of science is reinforced:

The mechanical clock extends the domain of quantification and measurability. Applying measure and number to time means measuring and quantifying all other areas, in particular those where time and space relate to one another. The measurability of time pushes forward the development of the natural sciences as (empirical) sciences of measurement (and hence objective sciences) and mathematics as the theory of measurement. (Keitel, 1989: 9)

Equally important, mathematics serves as the grammar of social order and social coordination. Keitel, Kotzmann and Skovsmose (1993) refer to F.W. Taylor’s (1947) introduction of "scientific management": Every complex work process can be broken down into elementary components; the time necessary to carry out these elementary components can be measured; the time in which a complex work process should be finished is the sum of the small but many 'pieces' of time needed for the elementary components. Here, the measurement of time “objectively” determines work organisation.

Keitel sums up:

Thus, the mechanical clock changed the relation between mankind and reality far beyond its original domain of application. It initiated the creation of a second nature totally reconstructing the first, exclusively admitting objective, mathematical laws, devaluing the authority of individual and collective (subjective) experience or insight. (Keitel, 1989: 9)

The "second nature" has replaced the first one and it appears as if mathematically conceptualised time were the most natural thing in the world. Indeed, nobody has to actually think about the consequences of this technological development or about the losses that its introduction has effected, when using a clock. The mathematical abstraction, which is encapsulated in it, has vanished from the surface – but it nevertheless continues to be effective:

*Implicit mathematics* makes mathematics disappear from ordinary social practice. (Keitel, 1989: 10)
Technology can be characterised by its effect of making the underlying (mathematical) abstraction processes invisible. At the same time, technology facilitates the use of mathematics in social or technical situations precisely by liberating the user from the details of the mathematics involved. A curious correlation can be observed: Whereas the flexibility and potential of mathematical thought lies in its harmlessness – there is no immediate threat of changing the physical world by carrying out mathematical abstractions and calculations, “we can handle hypothetical states of affairs” (Keitel, Kotzmann, & Skovsmose, 1993: 250) – the materialised mathematics of technologies has lost its innocence.

The abstraction encapsulated in many technological artefacts brings about flexibility in their range of application. It is exactly this flexibility that makes them suspicious. Technology that has been developed with the help of mathematical thought for a particular purpose, is often “flexibly” applied to new situations where new mathematical thinking might have resulted in a qualitatively different technological solution if a new and different mathematisation had been introduced. Consequently, the effect of this transfer of technology is restricting the scope of problem solutions that could have been imagined. Thus, a restrictive effect can be attributed to technology (as realised abstraction), whereas the mathematics (as thinking abstraction) offers hypothetical explorations into new problem solutions.

Mathematisation/ demathematisation as a shift of values

Fischer (1993) draws on the sociologist Tenbruck’s (1975) distinction of the value of meaning and the value of utilisation of scientific knowledge. The value of utilisation refers to the instrumental facet of scientific propositions, laws and procedures. These can be used efficiently as a means for solving social or technological problems. The value of meaning is associated with people’s need and desire to understand human existence and to gain orientation for life in society. As Fischer points out:

Value of meaning and value of utilization are not solely determined by the content of a scientific proposition. They depend on societal conditions, on already existing knowledge, and so forth. (Fischer, 1993: 115)

According to Tenbruck (1975), over time, scientific knowledge is subject to a shift of value. In the moment of its invention, scientific knowledge is valued mostly for its explanatory power. It provides new orientation for thinking about human existence – but it rarely has instrumental value from the outset. After some time, as Tenbruck argues, the value of meaning decreases: the new knowledge is built into the common explanatory framework. It becomes a matter of fact. This process is accompanied by an increase in its value of utilisation, the instrumental potential of the new knowledge is explored.

Fischer’s (1993) interpretation of Tenbruck’s thesis focuses on the development of mathematical knowledge. The value of meaning of mathematical knowledge was presumably at its height during the late 18th and 19th century when science and mathematics developed into separated organisations of knowledge with their
own respective fields of enquiry. At that time, mathematics was increasingly viewed as going beyond science, as a universal method for explaining the world. It stimulated a new approach to language and logic and to describing social behaviour by means of mathematical concepts and laws. In short, during the period of the foundation of 20th century mathematics, its value of meaning had substantially increased.

Nowadays, in contrast, Albert Einstein’s well-known saying that the closer a mathematical statement is to reality, the less certain it is, and that every reliable mathematical statement is far from reality, has become almost commonplace. A relativistic perspective is prevalent: we know that the connections between abstract and “sterile” mathematics and complex reality are not very tight. As Fischer (1993) puts it:

One does not expect that mathematically invented propositions are fully correct in reality. … Mathematics is often not used to describe reality, but to construct a new reality. For this it is excellent. (Fischer, 1993: 118)

The value of utilisation has replaced the value of meaning of mathematics. In order to illustrate that claim, Fischer (1993) discusses the highly interesting and relevant example of economic and financial policy. It does not make sense to take mathematical descriptions of trade balances, tax systems, stockbroking etc. as more or less adequate reflections of the financial reality. They actually construct this reality by setting up mathematical models that make rational financial decisions possible. Within this mathematically constructed reality, mathematics is automatically the best means for refinement of the constructions:

We have a circularity: The more mathematics is used to construct a reality, the better it can be applied to describe and handle exactly that reality. (Fischer, 1993: 118)

In the case of mathematics, the shift from the value of meaning towards the value of utilisation is often paralleled by a process in which mathematical knowledge materialises into technologies (social technologies included), thus becoming implicit knowledge. Demathematisation, in this context, denotes the phenomenon of mathematics having lost its value of meaning for the majority of people. For most individuals mathematics is of no help for existential questions. At the same time the value of utilisation is obscured – mathematics has become invisible: not even for using all the mathematics-based technology it is necessary to be mathematically competent. More than ever, however, people make use of “crystallized” mathematical knowledge, but instrumental mathematical competence is not required.

Bishop, in this volume, reports on a study of mathematics and science teachers’ values based on a model comprising of six sets of value clusters that are structured as three complementary pairs. The dimension of “control and progress” describes how individuals relate to their mathematical and scientific knowledge. These values resemble the ways in which individuals interpret the value of utilisation.
It can be argued, as Fischer (1993) does, that the value of utilisation, obscured in and through technology, is well connected to the value of meaning. The mathematical construction of many parts of reality constrains the ways in which people give meaning to their life. Following this line of argument, mathematics still is meaningful for people, but not in an explicit sense. Mathematics, in Fischer’s (1993) terms, is a means of control and manipulation of the natural and the artificial environment and a system of concepts and rules, embodied in our thinking and doing. While mathematics, as a means, is conceptually similar to the value of utilisation, the system aspect goes beyond the value of meaning. The system aspect has explanatory power with respect to the relation of mathematisation and demathematisation. It covers the fact that mathematics has transformed into a system that:

- runs from everyday quantifications to elaborated patterns of natural phenomena to complex mechanisms of the modern economy. … [This is a system] we have to obey and which is inseparably connected with our social organization. (Fischer, 1993: 114).

The value of meaning can be regarded as the conscious part of mathematics as a system. The unconscious part, however, is at the core of the relation of mathematisation and demathematisation. Mathematics has the potential of contributing to more consciousness of society, if it were not used dogmatically for legitimating the given, but would instead be interpreted as a means for provoking decisions, which might lead to change, as Fischer argues in this volume.

An important but often neglected aspect is illuminated by Davis in this volume. He presents an analysis of the public image of mathematics conveyed by the media. It shows that the value of utilisation is obscured and the public image consists of an imperceptive view of the fact that mathematisation “is changing what it means to be human”; mathematics is mystified by concentration on the sensational. As Ahmed, in this volume, argues, students in school usually gain little experience of the power of mathematics, not even of the “sensational”. They are likely to develop a narrow, techniques and convention oriented view of the subject. Malcolm, in this volume, makes a related observation of the way science is taught at school. “School science, traditionally, has taken the Enlightenment position, promoting and demonstrating reason, empiricism, objectivity and mathematical logic as the key characteristics of science.” Restricting school science to these values is one of the reasons for exclusion of distinct groups of students and thus counteracting the vision of 'Science for All'.

Mathematisation/ demathematisation and power

Skovsmose (1998) regards mathematics as an essential instrument for exercising technological power. He sees an increase in the range of applications of mathematics linked to modern information technology. Mathematics has not only become an integrated part of technological planning and decision making but also an invisible part of social structuration, encapsulated in political arguments,
technologies and administrative routines. Citizenship presupposes the excavation of “frozen” mathematics.

Following this line of argument, demathematisation excludes citizenship, and development of appropriate excavation tools becomes a central issue. Skovsmose introduces positions of social groups who are involved in or affected by mathematics in action in different ways. The “constructors” are those who “develop and maintain the apparatus of reason” (Skovsmose, 2006: 140). In constructing mathematics based technology, this group exercises power over “operators” and “consumers” of this technology. Vithal, in this volume, discusses issues of ethics and political responsibility and of social and cultural awareness of those who occupy positions of power. This affects the mathematics and science curricula of graduates as well as the opportunities of researchers. However, the so called “math wars” in the United States show, as Kilpatrick reports in this volume, that mathematicians oppose the standards developed by the Nation Council of Teachers of Mathematics – these standards being an attempt of providing a vision of mathematical literacy for today’s world. Their worries are, for example, directed towards “schoolchildren wasting their time making histograms of data they have gathered when they could be learning arithmetic. They see statistics in school mathematics as involving little serious work.”

Whereas the constructors are involved in developing mathematical technology, the operators are those who work in jobs, in which they have to make decisions on the input and then decisions based on the output of this technology. These job situations can be called “rich in implicit mathematics” (Skovsmose, 2006: 142). Operators are not only prepared for their tasks in terms of the content of their mathematical training, but also accustomed to the “habit” of following rules by the hidden curriculum of school mathematics. Skovsmose (in press) calls those who are listening to a range of offers, statements and reports containing figures and numbers, slightly ironically, “consumers” of mathematics. They could “vote, receive services, fulfil obligations, be citizens”. Consumers are confronted with justifications of decisions based on complex models. While the consumers can be regarded as “targets” of mathematisations, there is still another group of people Skovsmose calls the “ ‘disposable’”, those who are marginalised and who “do not define themselves with reference towards what they miss in a relationship with the globalised world” (Skovsmose, in press).

There is a threat to democracy because of a widening gap of mathematical knowledge between constructors and consumers. The constructors not only provide the technical knowledge for developing solutions but also have the power to define the problems and to initiate new questions. The forming of opinions and political decisions become more and more dependant on their expertise.

Skovsmose (1994) sees as one of the essential problems of democracy in a highly technological society, the development of a critical competence, which can match the actual social and technological development. If the interpretation of democracy is not restricted to formal procedures of electing a body of representatives, but also includes participation and elements of direct democracy,
the status of the constructors has to be scrutinised. Decisions made on the ground of mathematical models may be inaccessible to demathematised consumers.

Citizenship does not only imply being ready to live in and to face the “output” from authorities. It also means providing an “input” to authority, a “talking back” to authority. (Skovsmose, 1998: 198-99).

As to mathematisations, 'talking back' presupposes a wider horizon of interpretations and pre-understandings of mathematical knowledge than passive consumption of offers, statements and reports:

The fundamental thesis relating technological and reflective knowledge is that technological knowledge itself is insufficient for predicting and analysing the results and consequences of its own production; reflections building upon different competencies are needed. The competence in constructing a car is not adequate for the evaluation of the social consequences of car production. (Skovsmose, 1994: 99).

Skovsmose (1998) identifies three groups of questions related to reflective knowledge, which focus (i) on the relationship between mathematics and an extra-mathematical reality, (ii) on mathematical concepts and algorithms, and (iii) on the social context of modelling and its implications in terms of power.

The distinction between technological and reflective knowledge, which also resembles the distinction between operators and critical consumers (in opposition to demathematised consumers), is fruitful, but still has to be further elaborated with respect to its consequences for a conceptualisation of content and forms of mathematics education. Especially with respect to those groups of people who are deprived of any kind of formal education (the ‘disposable’), the tension between functional and critical education seems to be exacerbated.

Skovsmose, in this volume, asks in which ways mathematics, which might be conceived as important for their future lives by some students, but not necessarily by those who are already marginalised, could offer new opportunities. Mathematics education should not only try to relate to the students’ background, but to their foreground, that is the opportunities provided by the social, political and cultural situation as conceived by a student. Also, the discourse of learning obstacles is linked to students’ (social and cultural) background rather than to their foreground, which shows the political nature of the concept.

REMARK

This introductory chapter has reviewed a selection of literature, which explicitly refers to mathematisation/ demathematisation and discusses this phenomenon as a social process. An attempt has been made to draw out issues related to the formatting power of mathematics and its role as implicit knowledge, resulting in a process of demathematisation – a concept that, after having received considerable attention, is now threatened to be eclipsed by the proliferation of a discussion of
school mathematics, which shows a tendency of cutting off its own philosophical and political roots.

NOTES

i This example has been adapted from the WebPage of the King’s College of the University of London: http://www.kcl.ac.uk/phpnews/wnview.php?ArtID=661

ii Another example of technology Keitel repeatedly uses in order to illustrate how the development and transfer of technology results in demathematisation is the economic instrument of double-entry bookkeeping (e.g., Keitel, 1989; Keitel, Kotzmann, & Skovsmose, 1993; Damerow, Keitel, Elwitz, & Zimmer 1974):

Because of its enormous diffusion the double-entry model penetrates practically all fields of social practice related to money. And like crystallisation in a liquid, which starting from one point expands over the whole surface, the calculation model sets going systematization and formalization all over the area where it is applied. In industrial enterprises where for a long time production followed its own traditional patterns, “scientific management” and “system analysis” ultimately led to the restructuring of all production processes in the most minute detail towards the goal of systematization and standardization in order to bring them under the control of the calculation model and its prescriptions. (Keitel, 1989: 11-12).

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INTRODUCTION

Reflections on numbers and their properties led already in antiquity to the belief that propositions concerning numbers have a special status, since their truth is dependent neither on empirical experience nor on historical circumstances. In a historical tradition extending from the Pythagorean through the Platonic tradition of Antiquity, Late Antiquity and the Middle Ages, further through the rationalism and the critical idealism of Kantian and neo-Kantian philosophy to the logical positivism and constructivism of the present, this belief has been considered proof that there are objects of which we can gain knowledge a priori. Like a recurring leitmotif, the conviction that numbers are by nature ahistorical and universal is woven through the history of philosophy. A variety of reasons have been proposed to explain this puzzling phenomenon. The historian, on the other hand, is confronted with the fact that numerical techniques and arithmetic insights have a history that is, at least on its surface, in no way different from other achievements of our culture. In view of the variety of historically documented arithmetical techniques, it is scarcely possible to dismiss the assumption that the concept of number – in the same way as most structures of human cognition – is subject to historical development, which in the course of history exposes it to substantial change.

If we ask which of the two alternatives is correct or if, possibly, the alternative itself has to be called into question, we must remember that this is a matter not considered solved in either of the disciplines dealing with the number concept, but rather, on the contrary, one which has for some time been the object of fundamental and ongoing controversy. This is particularly true in the case of psychology.

In the initial phase of modern psychology under the influence of neo-Kantianism, numbers and similar mathematically determined objects were primarily regarded as results of thought processes found in all humans alike. Only with the reaction of Gestalt psychology (Wertheimer, 1925) to the challenge of universal concepts of number by cultural anthropology (Lévy-Bruhl, 1926) did the question of the nature of numbers find its way into psychology. In particular the empirical evidence, provided by Piaget, that the concept of number is not already...
imprinted in a child at birth, but is rather formed during the development of the child in a number of developmental stages (Piaget, 1952), contributed to undermining the belief in the innate nature of the concept of number. Piaget himself, however, still interpreted his results entirely in the spirit of neo-Kantianism. In his theory the development of the concept of number in ontogenesis rests on experience; however, the result of the development, according to this theory, is determined epigenetically – similar to many other biologically determined characteristics of humans. Numbers are thus considered as a cross-culturally universal cognitive construction that only appears at the end of the development of the individual (Piaget, 1950, 1971). Ethnographic research, conducted in the tradition of the developmental psychology that reflects Piaget’s work and methods, has predominantly used this premise as its point of departure, and has accordingly arrived for the most part at the conclusion that the speed of the development of logico-mathematical thought varies markedly under diverse sets of cultural circumstances, not so however the structure of the logico-mathematically structured concepts themselves (Piaget & Garcia, 1989; Bruner, Oliver, & Greenfield, 1966; Dasen & Heron, 1981; Dasen & Ribaupierre, 1988; Hallpike, 1979). Piaget offered a number of arguments suggesting that cognition in primitive cultures that have no developed arithmetical activities is comparable to the pre-logical stage of the ontogenetic development of a child, in which cognition cannot yet make use of the mental operations that are characteristic of the subsequent concrete operational stage of ontogenesis (Piaget 1950, vol. 2: 72-82). He distinguished two fundamentally different phases of development for each logico-mathematical concept: an initial phase in which the historical development passes through universal stages that are ontogenetically identifiable, and a second one in which the development is no longer subject to universal laws, but rather to a historical logic of development constituted by reflective abstractions (Damerow, 1995: 299-370).

Contrary to such universalistic interpretations of the concept of number, the examination of the particular mental processes connected with arithmetic operations led to theoretical approaches in which the emergence of numbers appears as the result of manifold learning processes (Gelman & Gallistel, 1978; Brainerd, 1979; Fuson & Hall, 1983; Smith, Greeno, & Vitolo, 1989; Gallistel & Rochel, 1992). Modern cognitive science has increasingly supported this view, recently by providing evidence that many arithmetic accomplishments can be attributed to the construction of relatively simple “mental models”. Further alternatives came into the discussion through the work of psycholinguists and their interpretation of the concept of number as a linguistic phenomenon, without, however, bringing the question closer to resolution. Under the influence of Chomsky’s theory, numbers have been ascribed to a biologically determined syntax of language (Hurford, 1975, 1985). Psycholinguistic investigations of the representation of logico-mathematical structures in language on the other hand suggest that we might understand such structures and the objects constituted by them better as culturally relativistic constructions.
Such contradictions between different conceptions of number obviously cannot be solved within the limited point of view of a single discipline, since neither a study of the cognitive functions of the concept of number excluding the question of its historical changes, nor a study of the historical development of arithmetical techniques leaving out of consideration the cognitive functions of those techniques, do justice to the unsolved problems that are revealed in these controversies. In what sense does the concept of number represent a universal? In what respect is it subject to historical changes? What implications result for the relation of the ontogenetic development of the concept of number to the historical changes of numerical techniques and arithmetical insights? These questions can only be answered by a historical epistemology of arithmetical thought that is compatible with psychological theories as well as with the results of historical research (Damerow, 1998). This view of the problems determines the theoretical program to be outlined in the following.

PRINCIPLES OF A HISTORICAL EPISTEMOLOGY OF LOGICO-MATHEMATICAL THOUGHT

On the nature of historical developmental processes of cognition

Since the historical development of cognition is realized through the cognitive activity of individuals, the description of cognitive abilities in the study of their historical development cannot, in principle, differ from that in the study of their individual development. Problematical, however, is the transfer of psychological concepts to historical development in the case of the developmental processes, insofar as individual development of cognitive knowledge representation structures is a process fundamentally different from their historical development.

The individual development of cognition is a process in the mind of the individual person. It starts with the awakening of intelligence in childhood and ends with the death of the person. The historical development of cognition, however, is a collective process spanning populations and generations, based on the interaction of individuals whose minds are basically independent of each other. The process of transmitting cognitive knowledge representation structures from one generation to the next takes place in a network of individual paths of tradition, leading from the individuals of one generation to the individuals of the next. These transmissions are realized in symbolic, and in part also in direct, interactions. There are no obvious reasons to assume that the network of those paths of tradition might show analogies to the individual development of cognition. The historical development of knowledge representation structures is by its nature a phenomenon that has to be interpreted socio-historically and not psychologically.

Nonetheless, the historical development of such structures is based on interactive processes, founded on very particular conditions that can be described psychologically. Not every individual process of achieving and structuring knowledge influences the historical development of cognition. Results of individual cognitive processes that are not transferable mental models which can be adopted by others in socialization processes, are obviously largely irrelevant to
historical developments. Likewise, the results of universal ontogenetic processes of development naturally cannot exhibit historical changes that might lead to coherent lines of development in the paths of tradition constituted by interactions. The network of those paths of tradition of cognitive abilities can apparently only then be subject to coherent processes of development when, in social interaction, results of individual cognitive processes are systematically reproduced and extended by consecutive generations. The reproduction of culture-specific mental models is therefore the most important psychologically describable condition for historical processes of cognition.

Basic assumptions concerning the development of logico-mathematical thought

The theory of the historical development of arithmetical thought proposed here is essentially based on two assumptions: Firstly, it is assumed, following Piaget’s genetic epistemology, that logico-mathematical concepts are abstracted not directly from the objects of cognition, but from the coordination of the actions that they are applied to and by which they are somehow transformed. According to this assumption the emergence of mental operations of logico-mathematical thought is based on the internalisation of systems of real actions. The internalised actions are the starting-point for meta-cognitive constructions, through which they become elements of systems of reversible mental transformations which, following Piaget’s terminology, we will call here ‘operations’. Meta-cognitive constructs such as the concept of number that are generated by reflective abstractions can thus be understood as internally represented invariables of mental operations which reflect actions on real objects. This explains the puzzling a priori nature of constructions such as the number concept. The experience of objects appears to be preformed by logico-mathematical structures. These structures, although they are subject to processes of development that have their origin lastly in the experience of objects, can therefore no longer be changed by those experiences.

Secondly, differing from Piaget’s theory, it is presumed that the basic structures of logico-mathematical thought are not determined epigenetically, but are developed by the individual growing up in confrontation with culture-specific challenges and constraints under which the systems of action have to be internalised. The challenges are embodied in the material means of goal-oriented or symbolic actions that are shared external representations of logico-mathematical structures of mental models. These mental models, upon which logico-mathematical competence is based, are thus construed in ontogenesis conditioned by the processes of socialization and by the co-construction of such models by means of interaction and communication. These co-constructions on the micro level of social interaction make it possible that mental models are transferred from one individual to the next and so, on the macro level of social development, are transmitted as intersubjectively shared schemata of interpretation. This process gains historical continuity through collective external representations which embody both mental models and levels of reflection, and thus levels of abstraction.
These two assumptions specify preconditions for a theory of the historical development of logico-mathematical thought. By combining both assumptions, a twofold result is achieved. On the one hand, certain objects of psychological theories receive a historical interpretation; on the other, historical stages of the development of thought can be characterized psychologically. The stages of the historical development of logico-mathematical thought can be interpreted as subsequent meta-cognitive levels that are connected with each other by reflective abstractions.

On the definition of historical stages of development

The application of psychological concepts to historical processes raises, however, a number of fundamental problems. One of these problems concerns the definition of historical stages of the development of cognition in general. Since psychological definitions of abilities by their nature do not refer to collective subjects, psychologically defined abilities cannot readily characterize historical stages of development. They can only be attributed to an individual person, to members of a group, or to all members of the society in a particular historical situation, not however to the society as a whole. A definition of historical stages of development using psychological concepts seems scarcely possible without determining, with a certain arbitrariness, on which of these different distributions of competence the definition of a historical stage as a criterion of its realization is to be based.

The second of the basic assumptions formulated above offers, however, a solution for this problem (Damerow, 2000). If the historical development of cognitive knowledge representation structures is essentially based on their intersubjective communication and historical transmission by means of external representations of mental models, then the social distribution of the competence is of only secondary importance for this development. It then represents only a framing condition, determining above all the speed of development and the chances of realization for the cognitive potentials embodied in the representations. The historical stages of development, on the other hand, have to be defined primarily on the basis of analyses of such representations and their possible functions in the individual development of cognition.

The theory of the historical development of logico-mathematical thought outlined here is therefore not meant to explain primarily the outstanding achievements of individuals nor the social distribution of abilities, but rather the historically changing potentials of development of the individual subject under the conditions prevailing at the time. In particular, the level of development of arithmetical thought in the various cultural epochs is not being defined by the actual results of arithmetical thought, but by the arithmetical means and external representations of mental models that were available for the ontogenetic development of arithmetic abilities, so that these could in principle evolve.

The resulting theory of the historical development of knowledge representation structures can be applied as well to the analysis of the construction of new representations which result from outstanding individual achievement as also to the
application of such a representation by a specifically trained group, or even to the
general use of such a representation in a society whose educational system
communicates such use. Its application to the development of cognition can
explain various aspects of historical stages defined by representations with certain
cognitive characteristics. The theory allows to infer that at these stages certain
individual accomplishments become possible, certain professional qualifications
become reasonable and certain goals of education become generally
understandable.

The transition from one stage to the next

In the historical development of cognition, the transitions from one stage of the
development of certain knowledge representation structures to the next one can
occur in two fundamentally different forms, namely either by cultural exchange or
by culture-immanent processes of construction.

The diversity of cultures coexisting and interacting with each other usually
results in the adoption of representations that have shown themselves to be
effective tools of cognition in another culture. Such processes of transmission have
to be studied in order to find out which of the cognitive knowledge representation
structures found in many, or even in all cultures are biologically inherent in human
nature and which, on the other hand, are a result of a transfer of representations of
these structures to many or all existing cultures.

Determinative for defining historical stages of development in the processes of
the historical genesis of knowledge representation structures is, however, the
development of mental models by culture-immanent constructions. This form of
development is based first on individual cognitive achievements that lead to the
modification of existing representations and to the construction of new ones. These
representations become part of a culture by being embedded in existing paths of
tradition, so that they can be integrated into the process of reproduction in this
culture.

In both cases, interactive co-constructions and transmissions of mental models
from one individual to the next constitutes a necessary precondition for the
emergence of a new stage of development. The individual, creative achievement,
which historians of science tend to credit with a crucial role in the rise of new
forms of thought, is only a peripheral condition of this development, which is
rather determined by existing representations of knowledge and by contingent
historical circumstances. Any historical situation defines a space of potential
cognitive achievements which enables individual creative achievements and, at the
same time, imposes narrow limits on them. The question of how the meaning of
historically determined representations of knowledge can be reconstructed by an
individual reflecting on his activities in a given cultural setting becomes a
theoretical key question for the understanding of the historical development of
cognition.

Two kinds of representations of mental models will be distinguished that are
fundamentally different with regard to the level of reflection crucial for their
meaning. The former will be called first-order external representations, the latter second-, or more generally, higher-order external representations. The difference consists, briefly stated, in the fact that first-order external representations stand for real objects and actions, higher-order external representations, however, for ideas and mental activities. How can such a differentiation be theoretically specified?

First-order external representations

Definition: First-order external representations of mental models (or briefly: first-order representations) are material representations of real objects by symbols or by models composed of symbols and rules of transformation, with which essentially the same actions can be performed as with the real objects themselves.xii

Some simple examples may make this definition plausible. The identification of a concrete object with a name, a word or a sign is a first-order representation of the object. Identifying such objects and their properties as permanent and grouping them with other objects can be performed with such representations in the same way as with the objects themselves. Another example is provided by counters and similar symbolic counting aids which represent real objects in one-to-one correspondences. Words used for counting and similar symbols such as body numbers that can be arranged in temporal and spatial succession are first-order representations of ordinal structures such as sizes or intensities of properties as, for instance, the shades of a colour range. They can serve to perform comparisons on the level of symbolic representation in order to compare sizes of the represented objects or to determine maxima and minima in a given real setting. Constructions with compass, ruler and similar graphic instruments are first-order representations of geometrical configurations in the Euclidean plane that can be used for identifying locations of objects, spatial relations between them, and changes of such relations when these objects are moved. The cutting and pasting of areas performed in a suitable medium of geometrical representation is a first-order representation of the additive structure of real areas such as, for instance, cultivated fields that are assigned to owners, used to grow specific plants, or selected for the allocation of the water from an irrigation system.

The usefulness of first-order representations is based on the fact that actions can as a rule be performed much more easily with the symbols of the representation than with the real objects they represent, since they are not, to the same degree, subject to accidental restrictions characteristic of real situations. Obviously, real actions represented by such symbolic actions cannot be substituted by them. Joining two groups of tokens representing two flocks of sheep, for instance, cannot substitute joining the two flocks themselves. The representation of sheep by tokens rather serves as a tool for performing symbolic actions anticipating the results of real actions in order to plan and control them.

First-order representations, however, are not only significant for the execution of operations of existing mental models but also contribute to their construction by the internalisation of systems of actions. Unlike in the case of their simple application to directly control real actions, symbolic actions in their role of
supporting the construction of mental models can completely substitute the real actions, because they share essential physical qualities with the objects and actions for which they stand. Symbolic actions in the system of rules of a first-order representation thus initiate the construction of the same mental model as actions with the real objects they represent. This quality of a system of actions will be designated in the following by the expression constructive. Symbolic actions that are performed with first-order representations are in this sense constructive with respect to the mental model that controls the real actions they represent.

First-order representations are abstract in the sense that the same symbols are used in diverse contexts. This leads to a differentiation in the meaning of symbols characteristic for this kind of representation. A symbol in a first-order representation embodies a concrete object which changes from application to application and, at the same time, an abstract object implicitly defined by the operations of the constructed mental model that remains the same in all applications.

Second- (or higher-) order external representations

Definition: Second- (or higher-) order external representations (or simply second-order representations) are material representations of mental models. They consist of symbols or models composed of symbols and rules of transformation which correspond to the operations of the abstract mental model that controls the actions performed with the real objects.

The adequate application of a second- (or higher-) order representation requires that it gets related to the real objects and actions to be represented by its symbols and transformations. This happens by assimilating these objects and actions to the mental model that gives the external representation its meaning.

Again, some simple examples may make this definition plausible. Words that are conventionally assigned to numbers (one, two, three, …) and non-constructive numerals (1, 2, 3, …) become second-order representations of abstract numbers once the developmental stage has been reached at which this concept occurs. Their application to real objects requires thus the understanding of the concept of number. The use of the word “number” itself and of any general terminology related to the attributes of numbers, the use of variables as universal numbers, and even abstract calculations with numerals independent of concrete applications are also examples of the use of higher-order representations of the number concept. Their applicability is based on the reflective manipulation of numbers and their representations, for example on the correct use of substitution rules for variables. Furthermore, certain theorems of Euclid’s Elements such as the Pythagorean theorem are second-order representations of the metric structure of the Euclidean plane. Albeit propositions on geometric figures, they are independent of these figures insofar as the objects they relate to are no longer these concrete figures but rather “virtual” mathematical objects that are implicitly defined by axioms and definitions within a framework of deductive representation.
Second-order representations represent real objects and actions only indirectly. They are not constructive with regard to the mental model they represent since their adequate application to real objects and actions presupposes that this model exists already. They are, however, constructive with regard to meta-cognitive mental models that may be constructed as higher-order representations by reflective abstraction as far as they are directly related to material actions performed on the real objects and actions (signs and sign transformations) of first- (or lower-) order representations which are the basis of the mental models they represent.

This shows the precise connection between second- (and higher-) order representations with reflective abstractions which, according to the first assumption formulated above, create logico-mathematical concepts. Since the material symbols and symbolic actions of second- (and higher-) order representations can themselves become objects of cognition, they initiate precisely that kind of concept formation for which Piaget introduced the term ‘reflective abstraction’.

Second-order representations result in a similar way as first-order representations in a differentiation of the meanings of symbols. While first-order representations initiate the differentiation between the concrete objects of real actions and abstract objects of mental models, second-order representations initiate the differentiation between such abstract objects and those objects which are implicitly defined by the symbols and symbol transformations of this second-order representation, which are abstract objects of a meta-cognitive nature.xv

The historical change of cognitive functions of external representations

As a rule, representations change their function in the process of historical development as well as in individual cognitive development. In particular, higher-order representations develop from first-order representations.

All systems of counting, for example, were originally first-order representations. They mainly represented ordinal structures, as a rule by the temporal succession of a conventionally determined counting sequence. They represented cardinal structures only insofar as by creating in the process of counting a one-to-one correspondence of real objects with words or body parts they could also be used as a first-order representation to identify cardinal quantities. With the development of the number concept, more and more abstract numerical properties became incorporated into their meaning. They thus developed into second-order representations of abstract numerical entities and arithmetic rules. In particular, they now also represented structures like multiplication which have no immediate basis in the symbolic action of counting. When the abstract concept of number had finally developed, the words used in counting developed into the abstract infinite counting sequence, the meaning of which incorporated, step by step, all deduced abstract properties of numbers such as, for instance, the infinity of the sequence of prime numbers. The counting sequence thus became a higher-order representation of the abstract concept of natural numbers.
The change of the function of representations was in a similar way also characteristic of the development of geometry. The prehistory of deductive geometry was shaped by the use of drawings and later also of true-to-scale constructions as first-order representations of relationships in empirical space. Constructions were still playing an essential, though different role in Euclid’s *Elements*. The ancient version of Euclid’s geometry comprised not only theorems with the proofs of their truth, but also so-called problems consisting of constructions and proofs that the constructed figures possess the required properties. This duality of constructions and proofs in Euclid’s *Elements* indicates that figures still served here as first-order representations complementing the deductive second-order representation in written language. In later editions and revisions of Euclidean geometry, the problems degenerated into helpful but basically dispensable exercises. The required information was now derived from mental images of the real figures which thus developed into second-order representations of the geometrical relations represented by such generalized mental images. The apparent independence of the mental images from the real figures seemed to show the a priori nature of geometrical knowledge. The construction of real figures developed into a meta-cognitive tool of judgments about the epistemological function of Euclidean proofs. This finally led to the construction of non-Euclidean geometries and the modern formalism of proof techniques, thus disproving the alleged a priori nature of geometrical knowledge. Meta-mathematical proofs of the consistency of non-Euclidean concepts now used geometrical figures as Euclidean models of abstract structures, i.e., as higher-order geometrical representations.

The historical transmission of the meanings of external representations

Two basic assumptions have been introduced at the beginning of this paper which are constitutive for the theory presented here. First, it is assumed that logico-mathematical concepts are abstracted invariants of transformations, transformations which are realized by actions, and second, that those abstractions are historically transmitted by collective external representations. These processes come about in culture-specific, historically changing symbolic scenarios. The key question to be answered is the question of how the members of a community can reconstruct and further develop the meaning of such historically transmitted external representations.

Since the mere reconstruction of the meaning of first-order representations does not yet require specific cognitive structures of logico-mathematical thought, they play a special role: first-order representations are the starting-point of abstraction and thus to a certain extent determine the structure of the semantics of second- (and higher-) order representations generated by reflective abstraction. In the same way that first-order representations of real objects and actions initiate the construction of new mental models, they also make possible the reconstruction of those mental models which already exist historically and which give a specific meaning to culture-specific first-order representations.
Second- (and higher-) order representations, that is, external representations of such mental models, require, however, that these mental models exist already. They must have been constructed or reconstructed already by means of first-order or lower-order representations respectively. As soon as this condition is established second- (and higher-) order representations have the same function in the reconstruction of the reflected meaning of mental transformations as first-order representations have for the meaning of concrete logico-mathematical activities themselves.

The historical development of logico-mathematical thought is thus based, according to this theory, on two psychologically explicable processes, namely on the individual construction and on the ontogenetic reconstruction of the meanings of representations. The differentiation of the functions of first-order and higher-order external representations provides a powerful theoretical tool for the explanation of historical as well as individual processes in the development of logico-mathematical thought.

Three different processes and their combined effects have to be examined, the ontogenetic emergence of seemingly universal cognitive structures which are independent of culture-specific representations, the reconstruction of the meaning of first-order representations, and the reconstruction of second- (and higher-) order representations.

First, there are systems of actions which develop on the basis of the biological potentials of humans and can consequently be found in all cultures. Those systems contain, for example, actions such as the grasping of objects, the arranging of objects, the movement of objects or of the own body in space, etc. These most general human activities correspond to the universals of logico-mathematical thought. Such universals are the basis of supercultural mental models that are not subject to historical changes. xvi

Second, there are very general systems of actions that are not determined by the human biological potentials but include the use of simple tools which are used in all or almost all cultures. They are therefore subject to historical changes, and are not found in all cultures in the same way. Once these systems exist, they exhibit in their simplicity little if any fundamental variation and are therefore largely transmitted by the same kind of first-order representations. xviii To these systems of actions belong, for example, the techniques of counting which are manifest in almost all cultures.

Third, there are countless culture-specific systems of actions and higher-order representations that are of importance for the reconstruction of the historical development of logico-mathematical thought since without them the complex mental models of culture-specific forms of abstract thinking that they embody cannot evolve in ontogenesis. xix

How can it be explained that the mental models of logico-mathematical thought constructed and transmitted this way by means of external representations seem to be logically determined, resulting from knowledge a priori? These mental models apparently are no longer connected to the empirical knowledge that constituted the starting-point for their construction. If the theory of the function of higher-order
representations submitted here is correct, there is a simple explanation for this characteristic. The independence of implicitly defined objects may be interpreted as the immediate consequence of the relative independence of higher-order representations from the meaning of lower-order representations that are their object. The elements of the meta-cognitive structures they represent cannot be related directly to the objects of the mental models they reflect. Consequently, the relations between them which constitute a higher-order mental model cannot be empirically falsified so that they appear, in spite of being constructed on the basis of empirical experiences, as knowledge a priori.

A particular consequence of this independence is that the symbolic transformations of a higher-order external representation need no longer to have any meaning at the level of the first-order representation of the real objects and actions from which the construction process started. This consideration provides an explanation for one of the most peculiar phenomena to be found in the individual as well as in the historical development of mathematical thought: in the course of development, formal conclusions contradict more and more interpretations of mathematical concepts at a lower level of abstraction, until finally, at a sufficiently high level of abstraction, those contradictions become irrelevant. In light of the theory of the function of external representations presented here, this phenomenon appears as an inevitable consequence of reflective abstractions.

A well known example is provided by the continuous expansion of the concept of apparently “natural” number by the development of seemingly unreal constructs such as “negative”, “irrational”, “transcendental” and “imaginary” numbers. Those numbers can no longer be interpreted as numbers in the sense of the original representation of sets of objects by counters. As long as numbers are applied exclusively to the context of their origin in transformations performed with concrete sets of discrete objects, such numbers therefore appear to be artificial constructions without any real correlate. Once numbers are understood formally, that is, as implicitly defined entities, however, the unreal character of these objects disappears.

A similar example is provided by non-Euclidean geometries. Those geometries appear absurd and unthinkable as long as geometrical concepts are, as Euclid’s problems indicate, applied to finite figures constructed with compass and ruler. This application of the concepts makes the construction of Euclidean models of non-Euclidean geometries, completely familiar to mathematicians and theoretical physicists and astronomers today, appear as unmotivated reinterpretations of the basic concepts of Euclidean geometry. What is used as a convincing technique of proof at a meta-cognitive level, appears to common sense as a violation of the seemingly “natural” meaning of geometrical concepts.

**HISTORICAL STAGES OF THE DEVELOPMENT OF THE NUMBER CONCEPT**

Based on the principles of a historical epistemology of logico-mathematical thought developed in the first part of this paper, we will draw in the second part consequences concerning the development of the number concept. After the
description of basic arithmetical activities, stages of development will be defined that result from these activities through reflective abstraction.

Arithmetical activities as a basis of the concept of number

The arithmetical activities that have to be considered as a possible basis for the historical origin and development of numbers are closely related to fundamental structures of the number concept, that is, to the structures of the so-called “natural” numbers, the positive integers. First, these numbers are ordered in a linear sequence. Their use as ordinal numbers, that is, as a tool to identify orders, is based on this order. Second, these numbers can be interpreted as equivalence classes of finite quantities of discrete objects. They can therefore be used as cardinal numbers for the identification of quantities. Third, they exhibit two fundamental arithmetical operations, addition and multiplication. It is therefore possible to perform calculations with these numbers.

In order to be developed, these fundamental structures require certain systems of actions. To determine relations of order that are to be represented by numbers, comparisons have to be made. To determine quantities, one-to-one correspondences have to be constructed. The additive relationship between three numbers results from the combination of two sets of objects to form a third; the multiplicative relationship results from a repetition of the same action or from a reproduction of the same configuration of objects.

These actions of comparison, correspondence, combination and repetition, integrated into systems, will here be designated as arithmetical activities. They link the mental models of arithmetical thought to real objects and situations, thus constituting the basis for attributing quantitative values and relations to objects of empirical experiences. Their importance for the emergence of fundamental structures of numbers lies in the fact that they can be reasonably used independent of each other and, in particular, independent of any integration by a number concept. On the other hand, the symbolic representation of the results of performing comparisons between real objects, the construction of correspondences between them, and the realization of combinations and repetitions of them immediately leads to first-order representations which may guide their meaningful integration by reflective abstractions and the second-order representation of them in symbolic numerical systems.

Developmental stages of the number concept as meta-cognitive levels of reflection

The question of when, where and how such abstraction processes were realized in human history and what the actual outcome was is a question of historical research, but the general patterns such historical developments follow are determined by the nature of the underlying processes of reflection. At the outset any development of arithmetical activities is to a considerable degree based on cognitive universals independent of culture (Langer, 1980, 1986). Given the great cultural differences in the actual formation of even such fundamental arithmetical activities it is, however,
obvious that even such fundamental arithmetical activities are already of a
historical nature. They are not based on cognitive universals alone, but are also
constituted by certain culture-historical processes of transmission. If developmental
stages of arithmetical thought are therefore defined as reflective abstractions of
historically developing arithmetical activities, that is, as meta-cognitive levels
reflecting actions of comparison, correspondence, combination and repetition, this
definition does not anticipate the answer to the question concerning the degree to
which the concept of number is determined by cognitive universals. Rather the
definition provides an analytical tool to study the question in historical sources.

If the historical development is interpreted as a sequence of gradually attained
levels of abstraction reflecting arithmetical activities, the construction process must
proceed in the following way (Damerow, 1996). At first, basic cognitive constructs
of arithmetical thought are abstracted from the actions of arithmetical activities
themselves. At the next level, they are abstracted from symbolic actions with
symbol systems such as tally systems, body-counting, calculi, etc., which replace
the original objects of the arithmetical activities. At higher levels, they are finally
abstracted from the formal use of the written representation of logical conclusions
concerning properties of numbers and numerical relations. As a result of this
consideration, four main phases in the historical development of the number
concept can be distinguished.

Before the development of the concept of number, there must have been a
period characterized by the complete lack of arithmetical activities in the above
defined sense. The emergence of the number concept starts with the
development of arithmetical activities as part of the culturally transmitted
techniques. Their symbolic representation resulted in first-order representations of
these arithmetical activities in the form of concrete tools for the control of
quantities. A third phase is reached when cognitive constructs that originated from
the reflection of real and symbolic actions of the first stage were represented by
symbols, and culturally transmitted by means of those representations. Finally, in a
fourth phase, the mental models which emerge from performing transformations
with such symbols and the mental operations which constitute these models are
made explicit and written down. In this way, they become formal rules for logical
transformations coded in written language.

These phases can be interpreted as levels of reflective abstraction. Since the
degree of abstraction is not determined by the process of reflection, the last two
levels can be further subdivided according to this degree. The six resulting stages
of reflective abstraction of the concept of number from arithmetical activities will
be designated as:

- stage 0: pre-arithmetical quantification
- stage 1: proto-arithmetic
- stage 2a: symbol-based arithmetic with context-dependent symbol systems
- stage 2b: symbol-based arithmetic with abstract symbol systems
- stage 3a: theoretical arithmetic with deductions in natural language
- stage 3b: theoretical arithmetic with formal deductions
These theoretically postulated stages have to be validated by specifying their definitions and by identifying them historically, relating them to results of historical research in order to test the assumptions they are based on. For this purpose each stage will first be defined theoretically. Second, semiotic characteristics will be described that may serve as criteria for assigning arithmetical techniques of a cultural context to this particular stage of development. Third, some concrete historical examples will in each case illustrate the respective goals of research that result from the proposed theoretical model for a historical epistemology of the development of the concept of number.

Pre-arithmetical quantification (stage 0)

Definition: The level of pre-arithmetical quantification is a stage of development in which comparisons are the only arithmetical activities. Pre-arithmetical quantifications are based only on comparisons. They include neither the construction of correspondences as, for instance, by counting, nor the composition of quantities by arithmetical operations, for instance the construction of numbers through the repetition of units.

Semiotic characteristics of pre-arithmetical quantification. The most noticeable difference of the level of pre-arithmetical quantification from all levels of genuine arithmetical activities is the absence of socially transmitted standards that might serve as tools for the construction of one-to-one correspondences. On the pre-arithmetical level there are no structured counting sequences and no tally systems such as finger counting, counting notches, counting knots or calculi. No words, signs or other symbols are used that possess any arithmetical meaning. The language at this stage possesses terms for quantities, yet these are of exclusively qualitative nature. Insofar as there are any rudimentary words for numbers at all, these are not used for counting; rather, they are designation for special qualities, that is, designations for intuitively and holistically understood small quantities. The quantitative aspects of an object of cognition are not yet distinguished from its specific physical appearance and from implications of its quantitative aspects (Ferreira, 1997).

Historical identification of pre-arithmetical quantification. A historiogenetic theory of arithmetical thought has first to address the question as to which original conditions for the historical development of this thought pattern are already preconditioned by cognitive universals founded in human nature and therefore not subject to historical change. In particular, the question arises as to whether cultures ever existed that correspond to the definition of a pre-arithmetical level given here. The answer to this question ensues from the fact that non-literate cultures existed until recently, which used no counting techniques before their contact with Western cultural traditions. The definition of a pre-arithmetical level applies to such cultures. The best known examples are the Australian aborigines (Dixon, 1980: 107f; Blake, 1981: 33; numerous examples in Dixon & Blake: 1979ff) and the South American natives (Lévy-Bruhl, 1926: 181-184; Gnerre, 1986: 74). It is, however, difficult to identify such cultures with certainty today since even the most
remote primitive peoples have been in extended contact with modern civilization. Through trade, which often provided the first systematic contacts with Western culture, they quickly assimilated arithmetical activities and concepts and also changed quantifications in their own language, for example, by inventing new, indigenous words for counting and new arithmetical techniques (Dixon, 1986: 108; further, in particular, Saxe, 1982). Proof that many of these cultures did not possess developed counting procedures before their contact with Western culture is today only feasible through often speculative linguistic inferences. Moreover, the linguistic material can often only be collected by interviewing a few elderly informants, whose children and grandchildren attend public schools in order to learn reading, writing and arithmetic. The influence of such contacts in many cases invalidates the information that can be gained from the informants, in particular because it consists primarily in the adoption of arithmetical activities which simply change the semantics of existing terms.

Identifying with certainty the pre-arithmetic level in the development of arithmetical thought historically constitutes an even larger problem, since written sources usually do not reach far enough into the past even to gain linguistic material for an identification according to semiotic criteria. An important clue is provided by the fact that from periods before the Late Neolithic no objects or signs, for example counting notches or calculi, have been identified that might have served as tally systems, or might have had another kind of arithmetical function. It is true that Palaeolithic, Mesolithic and in particular Neolithic finds, especially of bones, often exhibit repeated signs such as regular patterns of notches, and that these have occasionally been interpreted to be representations of numbers, but such an interpretation can hardly be justified, since these sign repetitions lack the characteristic subdivision by counting levels that would be expected in signs with numerical meaning, and which is indeed present in all known real counting systems.

Proto-arithmetic (stage 1)

**Definition:** The level of proto-arithmetic is a stage of development in which first-order representations of quantities are constructed by means of one-to-one correspondences to standard sets of concrete objects or other symbols.

**Semiotic characteristics of proto-arithmetic.** The earliest genuine arithmetical activities historically attested are without exception based on objects themselves being represented by symbols, their quantity however by the repetition of these symbols. Symbols are the most simple tools for the construction of one-to-one correspondences that can be transmitted from generation to generation. Structured and standardized systems of symbols, from which standard amounts can be formed that are assigned to the quantities to be identified, are the oldest tools for the identification and control of quantities.

Since symbols, in order to represent quantities, may be repeated either temporally or spatially, in principle two different kinds of such simple standards
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for the representation of quantities can be distinguished. The former will here be
called counting sequences, the latter tallies.

A counting sequence in this sense is a standardized sequence of words or
symbolic actions which is assigned to the elements of a given set in a fixed
sequence, realized in time, a process that is generally called counting. Through the
process of counting, a linear (temporal) first-order representation of the ordinal
structure of such a given quantity is realized.

Tallies, on the other hand, are concrete objects such as signs, knots, notches or
calculi, that can be arranged and combined in a simple way for the purpose of
constructing correspondences. By assigning such objects to the elements of a given
set, a spatial first-order representation of the cardinal structure of a given quantity
is realized.

Thus, counting sequences and tallying systems are two different forms of first-
order representations of finite sets of objects, representing different aspects of their
quantity. Common to both forms are certain characteristic structures originating
from their function. The deliberate construction of correspondences initiates, in the
course of historical development, processes acting to extend their range of
application. Genuine counting sequences and tallying systems therefore exhibit two
typical structural patterns that are a consequence of their progressive improvement
in the direction of an infinite counting sequence. First, older counting limits are
preserved as inherent steps of the counting procedure. Second, counting limits are
systematically overcome by certain techniques.

The predominant procedure for extending the range of application of counting
sequences and tallying systems is, when a counting limit is reached, to start over
with counting in connection with a counting procedure of higher-order which
determines how often the primary procedure had been used. The repeated
application of such techniques of passing counting limits generates a hierarchically
structured symbol system which can, with this technique, be extended to virtually
any required order of magnitude.

The characteristic structures of genuine counting sequences and tallying systems
resulting from this procedure are important indicators of arithmetical activities that
have to be attributed to the proto-arithmetic level. On the basis of their
hierarchically organized structures, tools for the construction of one-to-one
correspondences that have served in the control of quantities can be identified even
when there are no historical sources that give us definite information concerning
their original purpose. Thus there is in principle no problem in distinguishing
cultures that have reached the proto-arithmetic level from cultures at the pre-
arithmetic level.

The proto-arithmetic level is distinguished from higher-order levels through the
absence of arithmetical procedures, that is, of symbolic transformations, that
correspond to arithmetical operations like addition and multiplication. Such
arithmetical procedures require that the symbols which are transformed according
to formal rules no longer represent the counted objects, but rather their quantity.
According to the definition of the proto-arithmetic level this level is precisely
characterized by the fact that symbolic transformations apply to representations of
objects and not to the representation of representations characteristic of higher-order levels (Lévy-Bruhl, 1926: 181-223; Gay & Cole, 1967; Hallpike, 1979: in particular 236-279).

**Historical identification of proto-arithmetical tools and techniques are known to us mainly from surviving non-literate cultures. Such cultures are to be assigned to the proto-arithmetical level if they actually developed arithmetical activities. They display a wide variety of counting sequences and tallying systems for the construction of correspondences, realized in all kinds of forms, whereas arithmetical techniques that are based on the symbolic representation of quantities and the numerical relations between them are encountered relatively seldom. Their proto-arithmetical tools are used almost exclusively for the identification of quantities, not, however, for symbolic transformations with the purpose of quantitative prediction of results from real interaction with sets of objects.

The study of the proto-arithmetical level in surviving non-literate cultures is of great significance for the reconstruction of the development of the concept of number insofar as there is almost no opportunity to study this stage of development in historical sources. The study of such surviving cultures provides hints for the interpretation of pre-literate period archaeological finds with possible arithmetical functions. Comparisons of various surviving non-literate cultures with regard to their stage of development in arithmetical thought suggest, in particular, that agricultural cultivation, animal husbandry and housekeeping connected with sedentariness provide social conditions that make proto-arithmetical techniques useful and their systematic transmission and development possible. It thus turns out to be likely that the historical process of cultural development arrived at the level of proto-arithmetical in the Late Neolithic and the Early Bronze Age.

A system of clay tokens apparently possessing proto-arithmetical functions has indeed been identified as one widely used in the Near East during that period, that is the time span from the beginning of sedentariness in the areas surrounding the Mesopotamian lowland plain and in the Nile valley around 8000 B.C. until the emergence of cities around 4000 B.C. Thousands of these tokens were found in excavations, especially in those of the Mesopotamian alluvial plain and the Persian highland. The oldest clay objects ascribed to these symbols are dated to the beginning of the 8th millennium (Schmandt-Besserat, 1992: 36f). Their identification as belonging to a tallying system is based on the fact that in the transitional phase to symbol-based arithmetic they were converted into in the numerical signs of the proto-cuneiform writing system developed around 3200 B.C.

**The transition from proto-arithmetical to symbol-based arithmetic**

Although surviving non-literate cultures provide detailed knowledge about proto-arithmetical techniques, the conclusions that can be drawn concerning archeological finds from historical time periods of proto-arithmetical are very limited. The first phase in the development of arithmetical thought about which we have detailed
information provided by historical sources from different geographical areas is the
subsequent phase of symbol-based arithmetic. One of these areas, however, offers
in addition important information for the understanding of the origin of arithmetic.
Recent studies of archaeological sources from the dawn of literacy in the Near East
have lead to the identification of peculiar forms of arithmetical activities which are,
in all likelihood, phenomena of the transition from proto-arithmetic to symbol-
based arithmetic which may be defined as an intermediate phase.

Definition: The transitional phase between proto-arithmetic and symbol-based
arithmetic is a historical phase during which, just as at the proto-arithmetic level,
exclusively first-order representations of quantities are constructed. These are,
however, in contrast to the proto-arithmetic level, represented in a developed
symbol system and not only by arithmetical activities such as counting and
tallying.

Semiotic characteristics of the transitional phase. The characteristic semiotic
feature of this transitional phase are complex symbol systems used as counting
units, whose numerical values, however, are not constant, but vary with the context
of their application. They do not represent context-independent fixed numerical
meanings, but rather units of counting and measurement of products whose
numerical relations are determined by the social context in which they are
standardized by conventions. They differ in particular from the counting and
tallying systems of the proto-arithmetic level in that they are already the subject of
genuinely symbolic transformations. The basis is formed by transformations that
still are first-order representations of real actions; that means they have to be
interpreted as representations of economic transactions or real administrative
activities. In addition, transformations occur that are not only symbolic
representations of real actions, but use the potential of symbol systems for
performing transformations with symbols the corresponding real actions of which
are useless or even impossible. The purpose of such transformations is merely to
get knowledge about their outcome.

Historical identification of the transitional phase. The transition from proto-
arithmetic to symbol-based arithmetic is obviously related to the invention of
writing. However, cuneiform writing is presently the only writing system with rich
archaeological findings reaching back to the period of the transition from proto-
arithmetic to symbol-based arithmetic. These findings include thousands of clay
tokens, numerous numerical tablets, and about 7000 proto-cuneiform and proto-
Elamite texts and text fragments, almost exclusively economical texts with records
of quantities exhibiting the semiotic characteristics described above.xxvi

The transition from proto-arithmetic to symbol-based arithmetic occurred during
a relatively short period of time. The starting-point consisted of a system of clay
tokens which was in use as a proto-arithmetic tool in the Neolithic period of the
entire Near East. In all likelihood, these tokens served as first-order representations
of counting and measuring units, probably constituting an administrative control
system for goods and products. Together with the emergence of Mesopotamian
cities and the beginnings of a form of state organization and the centralization in
the administration of estates during the second half of the 4th millennium, this
The administrative control system was fundamentally revised. The tokens were first complemented and later completely replaced by markings that were impressed onto the surface of sealed clay tablets, the so-called numerical tablets. Around 3200 B.C., the numerical notations on these tablets were supplemented with protocuneiform pictograms, thus constituting the so-called proto-cuneiform writing system of Mesopotamia. The tablets also achieved a more complex structure by combining several quantitative notations on one tablet, arranged according to their function. Moreover, such numerical entries were often totalled. The number of numerical signs increased to some 60 signs, each representing units of measurement and counting.

These numerical signs exhibit the peculiar semiotic property which suggest the definition of the transitional phase given here. Unlike all other developed systems of numerical notations known so far, most of the proto-cuneiform numerical signs were used to represent more than just one unit of counting or measurement, regardless of the quantities represented by these units, i.e., the signs had no fixed but context-dependent numerical values. For the whole period of some 200 years during which the proto-cuneiform writing system was used the sources show no attempt to attain unambiguous numerical values for these signs. Already the texts of the Early Dynastic period, the first period following the time of proto-cuneiform writing, exhibit, however, the typical semantic structures of the numerical signs of symbol-based arithmetic and show scarcely any traces of the peculiar semantics of numerical signs from the transitional period.

Symbol-based arithmetic (stage 2)

Definition: The level of symbol-based arithmetic is a stage of development in which second-order representations of quantities and arithmetical activities are constructed by the reflection of proto-arithmetical mental models and by their representation in a symbol system. This development produces systems of numerical signs, which are semiotically structured according to the mental models they represent. The level of symbol-based arithmetic can be further subdivided with regard to the degree of abstraction from the specific context in which the meaning of the symbols is generated by reflection, resulting in a sublevel of context-dependent symbol systems and a sublevel of abstract symbol systems.

Semiotic characteristics of symbol-based arithmetic. The striking characteristic of the level of symbol-based arithmetic is the emergence of complex systems of numerical symbols and of formal rules for their application. Two different forms of such systems fulfilling different purposes can be distinguished, namely numerical sign systems and calculation aids. The former are used predominantly for the recording of quantitative information, the latter for its processing.

The formal rules for dealing with numerical sign systems and calculation aids may relate directly to the symbols, and may thus be explicitly stated. They may, on the other hand, only emerge from the strict application of the symbols according to their formal meaning. Such implicit rules can be identified by the consequences of their application, because their application is strict in the sense that their
applicability is no longer dependent on contingent conditions of the actual application context in which they are transmitted.

At the level of symbol-based arithmetic, the formal rules of symbol transformations usually remain implicit. Although at the level of symbol-based arithmetic usually the rules for symbolic transformations are still only implicit rules which can, moreover, be of a specific nature for their respective different contexts of application, they represent the first genuine arithmetical techniques. They constitute a new level of arithmetical activities applied to objects which are no longer the original real objects of the area of application, but the symbols of numerical notation systems and calculation aids and their abstract meanings. Therefore, at this level the direct relationship of counting sequences and tallying systems to the real objects they represent becomes more and more obsolete and the repetition of symbols in first-order representations gradually loses its function of supporting the arithmetical activities. Numerical notations as second-order representations are reduced to standardized signs used as names for entities in a mental model.

Context-dependent and abstract symbol systems. Since numerical symbol systems characteristic of the level of symbol-based arithmetic are, due to the implicitness of their semiotic rules, at first still closely related to the meanings of the symbols in specific contexts of application, they are not necessarily from the outset applicable to arbitrary contents.

At the sublevel of context-dependent symbol systems quantities are already symbolically represented by second-order representations, but they still possess specific areas of application. A generalized system of notations into which all special forms of notations can be transformed and thus become standardized is still lacking. The rules of sign transformation remain intermingled with the specific meanings of the signs in their particular context of application and thus are rarely formal rules, that is, rules depending only on the form of the signs and sign combinations.

At the sublevel of abstract symbol systems there exists an arithmetical symbol system that is generally applicable, independent of the context of application. It allows the standardized representation of all quantities, into which all particular application-specific representations can be transformed. There are no external limits to its application. Consequently, with such symbol systems formal operations can be performed, that is, without reference to any specific interpretation in their actual context of application.

Such abstract semiotic constructs without canonical reference to specific contexts of real objects and actions make possible for the first time forms of cognitive processing of abstract ideas that may be interpreted as early forms of genuine mathematical thought. The constitution of meanings reflecting formal symbolic transformations leads to knowledge of entirely implicitly defined artificial objects, with at best metaphorical reference to real contexts of application. This knowledge can be acquired, represented and historically transmitted as a first body of mathematical knowledge. However, the knowledge of abstract objects that are only mentally constructible is at this level not yet integrated into deductive
systems with formal rules of inference. Thus the qualities of these abstract objects cannot yet be derived or substantiated with logical derivations. But mental operations with such objects created by reflection are for good reason usually considered an early form of mathematical thought.xxviii

Historical identification of symbol-based arithmetic. Most, if not all, advanced civilizations, in particular the Egyptian empire, the Mesopotamian city states, the Mediterranean cultures, the Chinese empire, the Central American cultures and the Inca culture, have, independent from each other, developed or adapted from other cultures systems of numerical symbols that exhibit the semiotic characteristics of the level of symbol-based arithmetic. The basic symbols of these systems were, the same as at the proto-arithmetical level, signs for units and not for numbers, but part of the systems were now complex symbol transformations, that is, arithmetical techniques such as the Egyptian calculation using unit fractions (Neugebauer, 1926: 137ff; Chace, 1927; Vogel, 1958-1959, vol. 1: 31-44; Gillings, 1972: 20ff; Damerow, 1995: 176-199), the sexagesimal arithmetical technique of the Babylonians (Neugebauer, 1934: 4ff; Vogel, 1958-1959, vol. 2: 15-35; Damerow, 1995: 199-261), the transformations of rod numerals on the Chinese counting board (Needham, 1959; Juschkewitsch, 1966: 12ff; Li & Dù, 1987: 3-19), the calendar calculations in the pre-Columbian culture of the Maya (Thompson, 1941, 1960: 51ff; Gaida & Tear, 1984; Closs, 1986; Damerow & Schmidt, 2004: 143-145, 163-167), or the technique of the use of knotted cords (quipú) as administrative tools by the Inca (Locke, 1923; Ascher & Ascher, 1971-1972; Scharlau & Münzel, 1986: 80-93).

In all of these cultures symbol-based arithmetic emerged in a context-dependent form. Administrative bureaucracies have always been the institutions that created those complex techniques for the transformation of numerical symbols characteristic of the level of symbol-based arithmetic, and the specific purposes for which these techniques were developed determined the structure of the symbolic transformations. Consequently, the early systems of symbol-based arithmetic exhibit a great variety of formal structures. With the exception of addition, which, as a trivial consequence of the representation of quantities by repetition of symbols, developed in all cultures in almost the same form, the arithmetical techniques of the early civilizations reflect for the most part the culture-specific differences of the areas of application for which the respective systems had been developed.

Historical identification of the transition from context-dependent to abstract symbol systems. The historical identification of the transition from the exclusive use of context-dependent symbol systems to the construction of a unified abstract symbol system raises considerable difficulties. Only the actual use of a seemingly abstract symbol system shows whether it serves as a tool for rendering different forms of context-dependent numerical notations in a unified form of representation. But the sources preserved from early civilizations are often too meagre to allow of a sufficiently precise assessment of the area of application of a symbol system that would make it possible to distinguish between a context-dependent and an abstract use of such a system. Moreover, numerical symbol systems exhibit such great differences with regard to the scale of their area of application that some, when
compared to the others, appear to be much more abstract systems. These are mostly the counting sequences, for counting procedures in contrast to measuring procedures are as a rule so object-neutral that fortuitous peculiarities of the counted objects can scarcely influence the counting procedure. They can therefore easily be misunderstood as indications of an abstract arithmetic which, in fact, may not yet have existed.

The distinction between a sublevel of context-specific numerical symbol systems and a sublevel of abstract systems has been proposed here in particular in view of the historical development of arithmetic in Babylonia. Since clay used in Mesopotamia as writing material is extremely durable an abundance of administrative and mathematical cuneiform texts survived which document the development of arithmetical techniques from the earliest beginnings of proto-arithmetic to the creation of an abstract symbol system of symbol-based arithmetic.

These sources demonstrate that from the moment of the emergence of writing around 3200 B.C. until the invention of the sexagesimal place value system about 2000 B.C., exclusively context-specific symbol systems were used (Damerow, 1995: 219-237). One of these systems was a sexagesimal system. It was already strictly structured at the moment of the emergence of writing, probably corresponding to a similar sequence of number words. This sexagesimal system had already a wide range of application. It was, however, not used for all objects. Its range of application was essentially restricted to counting discrete objects.

Only with the invention of the sexagesimal place value system around 2000 B.C. was an abstract system of numerical notations introduced that unified all forms of notations. Some 100,000 administrative documents survived from the last 100 years of the third millennium B.C., the period of the third dynasty of Ur, which provide detailed information about the accidental circumstances which triggered this invention and about the systematic tendency towards a unification of numerical notations as well. This tendency in combination with specific circumstances was the ultimate cause of the transition from context-dependent to abstract symbol-based arithmetic in Mesopotamia.

The historical context which provided the conditions for the invention of the sexagesimal place value system was the unification of the city states by the empire of the 3rd Dynasty of Ur and the building up of a centralized administration of all resources, products and services. Within this context highly specialized book-keeping procedures with specialized metrologies and corresponding numerical notation systems had to be unified in order to get standardized judgements about values as a precondition of the universal exchange of such resources. At first, different administrative units, depending on the focus of their tasks, used specific value standards into which all resources were converted, standards such as quantities of silver, fish, barley or human labour (Englund, 1990: in particular 18ff, 96ff, and 181ff). Finally, however, the abstract sexagesimal place value system was developed which had no longer any association with specific metrologies. Metrological tables were used to convert the traditional notations into notations of this new system.
The invention of the sexagesimal place value system in Babylonia had two far-reaching consequences for the development of arithmetic. First, it revolutionized the Babylonian arithmetical technique. The multitude of arithmetical procedures developed for the solution of specific problems was replaced by one set of algorithms, including in particular a uniform multiplication procedure (Damerow, 1995: in particular 246-258). Second, the invention of the sexagesimal place value system had the development of the so-called Babylonian mathematics as a consequence. Reflection on the arithmetical operations of the sexagesimal place value system formed the basis for the construction of a system of technical terms, canonical types of problems and abstract mental operations which was only metaphorically related to its areas of application. Operations with the sexagesimal place value system made it possible to solve such complex, but at the same time practically irrelevant problems as the computation of the sides of a field from its perimeter and its area, that is of the second-degree equation of identifying two unknown quantities from their sum and their product.\textsuperscript{xxxii}

This description of the process and the consequences of the transition from context-dependent to abstract symbol-based arithmetic is based on the exceptional case of Babylonia where an abundance of sources survived documenting this transition. It is, however, likely that the symbol-based arithmetic developed independently also in other cultures such that the ancient Egyptian and the ancient Chinese arithmetic developed in a similar way from context-dependent systems.

\textit{Concept-based arithmetic (stage 3)}

\textbf{Definition:} The level of concept-based arithmetic is a stage of development in which second and higher-order representations of symbolic actions with representations of symbol-based arithmetic are constructed by the reflection of mental models depending on symbol-based arithmetic and by representing these meta-cognitive models in the medium of written language. This process of reflection results in logically structured systems of arithmetical propositions that make their deductive derivation and proof possible. The cognitive constructs that are thus abstracted from sign systems of symbol-based arithmetic become progressively independent of their specific properties. The level of concept-based arithmetic can be further subdivided with regard to the degree of abstraction from the specific context in which the meaning of the sign systems is generated by reflection, resulting in a sublevel of deduction in natural language and a sublevel of formal deduction.

\textbf{Semiotic characteristics of concept-based arithmetic.} The most important semiotic characteristic of the level of concept-based arithmetic is the explication of general propositions concerning the properties of abstract numbers. Since such propositions can only be deduced, they are naturally embedded in a system of deductive relations which connect them with each other. Under certain conditions such deductive webs of relations can be linearised, that is, the propositions can be globally structured in a way that all circular arguments are removed and all propositions appear to be systematically deduced from but a few basic
propositions. Accordingly, it is customary following the Euclidean tradition to write such propositions down in a deductive order as a theory.

Theoretical concepts do not owe their structure directly to the arithmetical techniques that constitute them, but to the knowledge that can be gained by the application of those techniques. These concepts are consequently embedded in structures of argumentation, that is, in structures at a meta-level of reflection. At this level of reflection, the meanings of arithmetical concepts that were seemingly determined by the rules of symbol-based arithmetic, may again be subject to development and can, if necessary, be modified by considerations of suitability of a higher kind. The concept of prime number as it was handed down to us by book VII of Euclid’s Elements (Heath, 1956, vol. 2: 296-344), for example, does not have the same kind of immediate technical meaning as do technical terms of symbol-based arithmetic such as sum, factor or divisor. Only the reflection of all possible summations, multiplications, divisions, and their results that can be inferred in a deductive system, leads to the formation of such a concept as a prime number being a number that cannot be further decomposed into factors other than one and the number itself and to the derivation of propositions such as the proposition that every natural number can be unambiguously factored into a product of prime numbers.

The meanings of such concepts resulting from the reflection on mental models depending on symbol-based arithmetic differ considerably from the meaning of concepts based on the symbol-based arithmetic itself. While the latter get a justification from their practical applicability, the former are apparently independent of any applicability, seemingly being determined only by internal consistency of deductive relations. Abstract numbers appear to exist a priori, because they have their origin in reflection. Numerical notations that are used to handle them for practical purposes, for example, appear to have no influence on the truth of statements about their properties. On the other hand, new constructions become possible on a meta-level of reflection. Questions such as the question of whether numbers might be conceivable with properties differing from what can be derived under the given conditions turn out to be justified even if such imagined numbers have no meaningful interpretation any longer in the context of arithmetical activities from which the process of mental construction started.

**Deduction in natural language and formal deduction.** Similar to the level of symbol-based arithmetic, the level of concept-based arithmetic can, according to the degree of abstraction, be divided into two sublevels.

The sublevel of deduction in natural language is characterized by the fact that the deductive systems consist of statements and proofs that are formulated in natural language. Mathematical terms, for instance the concept of number, are explicitly defined and are abstract insofar as they are determined within the logical structure of a deductive system. They still refer, however, to concrete objects and actions, since a representation in natural language entails connotations of the concepts that are determined by their origin in real actions. Numbers, for example, have an abstract structure on the one hand, on the other a connection to the
arithmetical activities that the numerical notations are based on; they not only have provable also apply to quantities of real objects.

Arithmetical concepts at this sublevel of development thus have at the same time extrinsic meanings that result from their origin in arithmetical activities and intrinsic meanings that are deduced from seemingly self-evident axioms. This results in particular in a canonical meaning of the concept of number at this level of development, usually expressed by the term natural number. Natural numbers are extrinsically determined as reflectively constructed structures of actions of correspondence and comparison. As cardinal numbers of quantities, they have thus a canonical object that determines their quasi natural properties. Intrinsically, their properties are determined by universal laws such as the distributive law or the commutative law of addition and multiplication. Those laws can be arranged deductively in a way that makes them appear to be logical conclusions of a few, seemingly self-evident axioms, for example the Peano axioms.

At the sublevel of formal deduction, the concepts formulated in natural language are replaced by terms of formal languages, so that connotations with their original meanings are systematically avoided. The concepts can be subsumed under generalized, unifying concepts that may be constructed artificially, and be entirely determined by mathematical structures. Their meanings no longer appear to be determined naturally, but through axioms that seem to be presupposed arbitrarily.

Numbers at this level appear as superposition of algebraic and topological structures precisely defined by axioms. Such axioms can be modified and in various ways combined with each other into suitable new structures designated artificially as semigroups, groups, topological groups, rings, fields, etc. Numbers have no longer an exceptional status among these mathematical objects; their apparent “naturalness”, characteristic for the sublevel of deduction in natural language, appears to be an fortuitous historical relict.

Historical identification of concept-based arithmetic. According to the definition of the level of concept-based arithmetic, the transition to this level is not a process specific to arithmetic. It rather applies to the reflection of mental models representing symbolic actions in general, provided that the results are encoded in the medium of written language. The encoding is essential for the transition to a concept-based cognitive construct. The reflection of mental models representing symbolic actions is in itself an inherent outcome of any symbol-based activity. In the case of symbol-based arithmetic such processes of reflection led to the emergence of various forms of so-called pre-Greek mathematics in early civilizations such as ancient Egypt (Neugebauer, 1934; Gillings, 1972), Babylonia (Neugebauer, 1934; Friberg, 1990; Hoyrup, 2002), India (Joseph, 1992), and China (Li & Dù, 1987). Such systems were transmitted within a culture from one generation to the next primarily by fostering the building up of the culture-specific mental models through exercising the symbol-based activities which they represent, partly also by teaching explicit rules of how to solve problems by means of symbol-based arithmetic. However, the first examples of an explicit representation of chains of inferences, which are based on abstract mathematical
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objects and result in universal propositions about such mental entities, are known from ancient Greece.xxxiii

The starting point of the development of Greek mathematics was the explicit representation of properties of numbers implicitly embodied in abstract numerical symbol systems in the medium of natural language (Szabó, 1960, 1978, 1994). This medium at the same time preserves the relation of numerical concepts to their origins in arithmetical activities. Thus in the Greek tradition numbers appear to be logically determined and at the same time empirically valid with regard to the objects of these activities; this is made explicit in the Platonic concept of pre-existing ideas.

The oldest known example of a deductively ordered system of propositions concerning properties of abstract numbers is the so-called doctrine of even and odd numbers, a theory that has not come down to us in its original form through preserved sources, but which can be reconstructed from definitions and theorems of Euclid’s Elements.xxxiv The theory goes back to the Pythagoreans of the 5th century B.C., who tried to associate all objects with numbers (Becker, 1936; van der Waerden, 1947/1949, 1954: 96-97, 108-127; Heath, 1921, vol. 1: 67). Its propositions are mainly concerned with the dependence of the property of a calculated number to be either even or odd on the respective properties of the original numbers entered into the calculation. The 14 relevant theorems that have come down to us reflect experiences with geometrical configurations of Greek counters, the so-called figured numbers (van der Waerden, 1954: 98-100). In the modified form in which the theory has been transmitted by Euclid, the theorems appear to be arranged in a deductive schema (Lefèvre, 1981; Damerow & Schmidt, 2004: 167-173). This development of the theory from the symbolic representation of numbers by arrangements of counters to a representation in written language and deduction of their properties from basic assumptions marks precisely the transition from symbol-based to concept-based arithmetic in the sense of the definition proposed here.

Euclid (Heath, 1956, vol. 2: 277, 280) transmits in his Elements also a more comprehensive theory of numbers, based on the Platonicxxxv definition “number is a quantity composed of units.” But even this theory could cover only a small part of the range of existing symbolic actions of symbol-based arithmetic provided by pre-Greek mathematics. The concepts of Greek theoretical arithmetic compiled in Euclid’s Elements were incompatible with arithmetical activities which did not belong to its foundations. In particular, fractions could not be related to the theory of number based on counting sequences and tallying systems represented by the definition “number is a quantity composed of units,” from which the properties of numbers were deduced. Platonism dogmatically excluded from theoretical arithmetic all numerical structures incompatible with this definition.xxxvi For a long time to come, concept-based arithmetic coexisted with the much richer tradition of symbol-based arithmetic.

Historical identification of the transition to formal deduction. The further development of symbolic means to represent propositions derived from symbolic actions of symbol-based arithmetic produced formal structures of higher-order
levels of reflection, structures which were increasingly distant from the basic
arithmetical activities that underlay the Greek theory of numbers. Historically, this
process presents itself as a continuous process of the construction of numbers by
reflective abstractions which did not fit the ancient definition of number. As long
as the ancient concept of number determined thinking, they were perceived as
“absurd numbers,” “irrational numbers” and “imaginary numbers,” since they
were incompatible with the historical connotation of the term number to designate
attributes of concrete sets of objects consisting of enumerable units.

The precondition for overcoming this understanding of number and thus for the
transition to the level of formal deduction was the development of the analytic
tradition which provided techniques of designating “unknown” mathematical
objects such as numbers by using letters, in modern terminology “variables,”
instead of meaningful descriptions by words, and operating with them as if they
were the mathematical objects themselves. Such operations could be performed
strictly according to the symbolic actions on higher level representations of
symbolic arithmetic without any connotations with the original meaning of the
mathematical objects they represent as, for instance, the Platonic definition of
natural numbers.

Descartes and Leibniz were early advocates of the analytical method. This
method initiated in the course of the 18th century the creation of completely new
mathematical disciplines which for a long time competed with the “synthetic”
method of mathematical disciplines using the Euclidean style of deductive
mathematical reasoning. It was, however, not before the end of the end of the 19th
century that both traditions merged into the modern technique of formal proofs
using implicit definitions by systems of axioms to determine the nature of
mathematical objects independent of meanings derived from their historical
origins. All types of numbers were relegated to examples of such implicitly defined
mathematical objects.

Final remarks

Historical epistemology is supposed to answer the question of whether in the
history of knowledge systems stages of development can be identified which at
each time determine the range of possible knowledge achievements in individual
cognition. Accordingly, a historical epistemology of the development of the
concept of number aims at reconstructing the changing structure of arithmetical
thought which is represented in the diversity of arithmetical techniques studied by
historians of science.

The result of an epistemological investigation certainly cannot consist of
replacing historical arguments with theoretical arguments. Indeed, the theoretical
considerations in the first part of the work have by no means made historical
research redundant. The answer to the question, to what degree the structures and
processes of arithmetical thought represent culture-independent and historically
unalterable universals of the nature of homo sapiens, and what part of arithmetical
thought on the other hand derives ultimately from cultural achievements and what
structures of this cognition have developed in which historical periods, is by no means theoretically pre-empted by the considerations presented here. On the contrary, they provide a precondition to study and answer such questions by historical and cross-cultural studies, for the theory of the historical transformation of arithmetical knowledge by reflective abstractions presented here opens up an opportunity of reconstructing the nature of arithmetical thought under given historical conditions through an analysis of its external representations.

The brief outline of the historical development of arithmetic presented in the second part of the paper is thus not derived from the theory of the historical transformation of arithmetical knowledge presented in the first part, but rather the result of relating developmental stages of reflective abstraction to obvious differences between representations of arithmetical knowledge in different historical periods. This result can be briefly summarized in the following schema.

Stage 0  *Pre-arithmetical quantification:* approximately until the end of the Mesolithic period (in the Near East until ca. 10,000 B.C.).
No arithmetical activities. All judgments about quantities are based on direct comparisons of amounts and sizes. Communication and transmission only by transmittable techniques of comparison and by comparative expressions of language.

Stage 1  *Proto-arithmetic:* Neolithic period and Early Bronze Age (in the Near East until ca. 3000 B.C.).
Quantities are precisely identified by one-to-one correspondences. Communication and transmission with the aid of conventional counting sequences and tallying systems.

Stage 2a  *Symbol-based arithmetic with context-dependent symbol systems:* Period of the early city cultures (in the Near East until the invention of the sexagesimal place value system around 2000 B.C.). Quantities are structured by metrological systems. Communication and transmission of these systems and of the corresponding mental constructs through complex symbol systems and developed techniques for the transformation of symbol configurations.

Stage 2b  *Symbol-based arithmetic with context-independent symbol systems:* Period of developed city cultures (in the Near East until the beginning of Classical Antiquity around 500 B.C.).
Quantities are structured by abstract numerical systems with object-independent arithmetical operations. Communication and transmission of these systems by unified, context-independent, but culture-specific symbol systems for the representation of arbitrary quantities, including abstract “rules of calculation”. Emergence of first forms of “pre-Greek mathematics” that are abstract but dependent on culture-specific symbol systems.
Stage 3a Concept-based arithmetic with deductions in natural language:
Classical Antiquity, Late Antiquity, Middle Ages and Early Modern Era (until the emergence of analytical mathematics in the 18th century A.D.).
Abstract number concept with “a priori” provable properties. Communication and transmission with the aid of a written representation of “propositions” about abstract numbers and their mathematical properties. Propositions are logically ordered and systematically arranged by deductive theories according to the model of Euclid’s Elements.

Stage 3b Concept-based arithmetic with formal deductions: The modern mathematical tradition until the present.
Formal understanding of arithmetical structures and expansion of the number concept by construction of new arithmetical structures. Communication and transmission with the aid of formal language systems.

This schema is preliminary. It can be only the first step of realizing the aim of a historical epistemology of arithmetic to explain how arithmetic evolved from its roots in incipient arithmetical activities which survived in certain non-literate cultures to the abstract and highly complex arithmetical thought of modern mathematics. It provides, however, a theoretical framework which makes it possible to pursue historical research that reconstructs the development of arithmetical thought as an outcome of the changing material culture of calculation.

NOTES

See the diverse arguments for this “Platonic” view, for example, in transcendental idealism, particularly Kant’s arguments in his Prolegomena (Kant, 1985, §§6-13) and in his Critique of pure reason (Kant, 1998, A 154-158 = B 193-197: 281-283); in neo-Kantianism, particularly Cassirer’s arguments in his Philosophie der symbolischen Formen (Cassirer, 1953-1957, vol. 1, ch. 3, sect. 3, in particular: 238f, and vol. 3, part 3, ch. 3: in particular 341-356); in logical positivism, for example Frege’s arguments in The foundations of arithmetic (Frege, 1959), and Carnap’s arguments in his Foundations of logic and mathematics (Carnap, 1955); further, in constructivism, Lorenzen’s arguments in Einführung in die operative Logik und Mathematik (Lorenzen, 1955).

See, for example, the historical accounts of Tropfke (1930-1940, 1980), Menninger (1970), Gericke (1970), and Ifrah (2000).

In a number of studies such a parallelism of ontogenesis and historiogenesis is discussed from different viewpoints (Bachelard, 1994, 1996; Arcà, 1984; Strauss, 1988; Dux, 1992: in particular 23-35, 1994).
A “mental model” (Gentner & Stevens, 1983) is conceived of here as special type of “frame” (Minsky, 1975, 1985; Davis, 1984) that represents shared, historically transmitted knowledge. On the nature of arithmetical skills see also Ashcraft (1992) and Campbell (1992).

An intensive debate on culture-relativistic conceptions of cognition was provoked by Whorf (1956). On relativism in cognition see also Pinxten (1976) and Gumperz & Levinson (1996).

This aspect has been emphasized in particular by the culture-historical school of psychology (Vygotsky, 1987; Leontyew, 1981).

See Tomasello and Call (1997: in particular 136-161) on the question whether humans share such knowledge representation structures even with certain primates and Damerow (1998) and Tomasello and Call (1997: 423-427) on the phylogenesis of human cognition from the viewpoint of cognitive psychology.

A typical problem of this kind is, for example, the debate about whether universal structures of language originate from the spreading of a “proto-language” or whether in those structures universal, biologically founded cognitive structures express themselves (Bickerton, 1988; Renfrew, 1988; Bateman et al., 1990). Similar questions arise in the case of the transition to literacy (Damerow, 2006). Can writing systems originate from completely independent roots? Are there, in particular, completely independent inventions of systems of arithmetic symbols?

This and the definitions given in the following do not correspond with the terms introduced by Bruner for the characterization of external representations (Bruner, Olver, & Greenfield, 1966: 1-67). In particular, his classification into enactive, iconic and symbolic representations is not applied here, since, as will be discussed in the following, it appears unsuited to adequately conceptualise the reflective structure of representations and the reflective dynamic of the relationship between symbols and the objects they represent.

That the use of the term number can be seen as an indication of a higher level of meta-cognition compared to the simple use of designations of numbers is apparent from the fact that non-literate cultures, even those with developed systems of counting, do not as a rule possess terms of this kind. This is in keeping with the fact that in those cultures even abstract counting without identification of concrete objects of counting is often regarded as meaningless. Even the early high civilizations with developed mathematics, for instance Egypt and Babylonia, do not have a term corresponding to our word number.

The use of variables instead of specific designations of numbers offers the possibility to determine precisely the degree of generalization of statements and to change it by substitution of variables in a controlled way. It thus opens a potential means of representation for a higher level of meta-cognitive insights such as the recognition that the natural numbers can be complemented with negative numbers to the system of whole numbers.

A prominent example of such a transfer of the constructive character of a representation to a meta-cognitive level is offered by the emergence of the deductive method in Greek mathematics. At first, arithmetical insights were constructed by figured numbers, that is, patterns of geometrically arranged counters, and geometrical insights by constructions with compass and ruler. The definitions and rules of deductive systems (e.g. those of Euclid’s Elements) are no longer constructive in this sense, but they are constructive with regard to the structuring of the mathematical knowledge gained in the process.

The meta-constructive character of higher-order representations may, for example, be used to prove, through the construction of models, the relative consistency of a deductive system. The so-called “Klein model” is a Euclidean model of the hyperbolic, non-Euclidean geometry; it demonstrates that a contradiction in hyperbolic geometry would generate a contradiction in the Euclidean geometry. By means of meta-mathematical semiotic reflection, a geometric figure is here constructed which, contrary to Euclid’s constructions, serves only meta-cognitive purposes.

According to Piaget, the logico-mathematical operations of operative intelligence pre-exist as practical intelligence in the pre-operative phase of ontogenetic development in the form of sensor-motorical schemata of action. These “proto-logic” systems of action have been systematically examined, in particular by Langer (1980, 1986).
Such systems and their cognitive effects have been made the object of research in developmental psychology, in particular by Piaget; see (Piaget, 1950). To the extent that such systems (for instance, the technique of constructing one-to-one correspondences to a standard of counting) are indeed nearly universal (Pinxten, 1976; Dasen, 1977), they can be identified in surveys of the historical development of numbers (e.g. Menninger, 1970; Tropfke, 1930-1940; Ifrah, 2000) with relative ease.

All arithmetical skills that result from processes of learning are probably based on these culture-specific systems of action and representations (Campbell, 1992). Even though there are precious few cross-cultural studies concerning those skills (Ashcraft, 1992), there is no doubt that they are largely of culture-specific nature and depend on the arithmetical tools that are known to us from ethnomathematical and historical sources.

This listing should not be misunderstood as a paraphrase of an axiomatic representation but rather as a hint for identifying basic arithmetical activities which may be considered as candidates for the historical origins of numbers. The structure of natural numbers can be characterized axiomatically in various ways. Through the Peano axioms, numbers are reduced to the iteration of units. Their definition as equivalence classes of sets with equal cardinality characterizes them as a structure of one-to-one correspondences. In a definition as semigroup with specific characteristics natural numbers are characterized by arithmetical operations. Those reductions of the structure of natural numbers to a few basic conditions are of only limited interest for the reconstruction of the origin and development of numbers, for numbers have historically certainly not been deduced from an (axiomatically describable) original definition, but probably rather originated through the integration of structural elements of their overall structure.

For example the “Klein model” of hyperbolic geometry mentioned above.

See e.g. the detailed description of the cultural environment provided by Lancy (1983).

See e.g. the interpretation of notches on a bone excavated at Ishango by Heinzelin (1962), who assumed that the irregularity of the numerical pattern of the notches is an indication of especially sophisticated arithmetical knowledge including, for instance the knowledge of prime numbers. See further, in particular, Marshack’s (1972) interpretations of numerous findings. See also the controversy about Marshack’s interpretations, which is documented by the criticism of D’Errico (1989a) and the rejoinders of Marshack (1989) and D’Errico (1989b).

The discovery of this function is of recent date. The first archaeologists who interpreted these tokens as arithmetical representations are Amiet (1966) and Schmandt-Besserat (1977). Current literature on the functions of these clay tokens is extensive. Nevertheless, no convincing reconstruction of what objects and quantities they represent has been possible so far. To know what combinations of such tokens are hidden in numerous excavated but still unopened clay bullae might help to solve this problem (Damerow & Meinzer, 1995).


Although detailed studies of the sources have been presented, there still exists the occasional prejudice that the Greeks were the first to develop abstract mathematics and that all pre-Greek mathematics was oriented exclusively towards practical purposes. This view can no longer be supported considering the original sources that have come down to us (Heyrup & Damerow, 2001).

System S using the notation introduced by Damerow & Englund (1987); not to be confused with the later sexagesimal place value system.
The recorded Sumerian sequence of number words was strictly sexagesimally structured (Powell, 1973). Since this sequence of number words has only been preserved in very late sources, however, the possibility cannot be excluded that it was artificially created by the scribes of periods following that of the emergence of writing, based on sexagesimal numerical notations in the cuneiform texts available to them.

This date is based on archaeological evidence as well as on the consistency of the historical development of calculation techniques (Nissen, Damerow, & Englund, 1993; Damerow 1995: 199-261; Hoyrup & Damerow, 2001: 219-310; for the transition period see Robson, 1999). A different opinion about the dating is held by Powell (1976).

The so-called mathematical cuneiform texts mainly come from this period. Compare for this the standard literature on the history of mathematics, in particular the work of Neugebauer (1935-1937), Neugebauer & Sachs (1945), Friberg (1990), and Robson (1999). The interpretation of these texts has, however, been subjected to a substantial revision in recent years (Hoyrup & Damerow, 2001), in particular, due to new translations of the mathematical termini technici for arithmetical operations suggested by Hoyrup (1990, 2002). The view argued here of Babylonian mathematics resulting from the reflection of culture-specific arithmetical activities at a level of a unified abstract symbol system is based on such new findings.

A recurring topos of mathematical historiography states that pre-Greek mathematics did not know proofs (Becker & Hofman, 1951: 41; Vogel 1958-1959, vol. 2: 84f; Wussing, 1965: 56.; Gericke, 1970: 20, 1984: 71ff). Historians who study pre-Greek mathematics have with good reason repeatedly raised the objection that here a particular form of representing deductive conclusions is being confused with the mental operations themselves and thus that European mathematical tradition, of which this form of representation is characteristic, is being eurocentrically overestimated (Joseph, 1992; Gerdes, 1990: in particular 24ff; for Chinese mathematics: Chemla, 1991; for Babylonian mathematics: Friberg, 1990: in particular 582ff; Hoyrup, 1990). In fact, pre-Greek mathematics is distinguished from Greek mathematics not by a lack of deductive reasoning but rather in the way such reasoning is externally represented. Once, however, the steps of such reasoning are externally encoded in written language or symbols, a new type of symbolic activity on a meta-level is induced. Such a development makes Greek mathematics exceptional in comparison to other cultures at its time.

Several Greek mathematicians used this definition. The most explicit passage in Plato's work elucidating the underlying idea of number is the dialogue about arithmetic in his Republic (Plato, 1994, 524d-526c: 158-167).

In Plato's Republic Socrates instructs his interlocutor Glaucon: 'For you are doubtless aware that experts in this study, if anyone attempts to cut up the 'one' in argument, laugh at him and refuse to allow it; (...) Suppose now, Glaucon, someone were to ask them, 'My good friends, what numbers are these you are talking about, in which the one is such as you postulate, each unity equal to every other without the slightest difference and admitting no division into parts?' What do you think would be the answer?'" “This, I think—that they are speaking of units which can only be conceived by thought, and which it is not possible to deal with in any other way.” (Plato, 1994, 526a: 164-167).

Michael Stifel called the negative numbers “numeri absurdi” (Tropfke, 1930-1940, vol. 2: 98).

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