Handbook of Research on the Psychology of Mathematics Education
Past, Present and Future

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Past, Present and Future

Edited by

Angel Gutiérrez

Universidad de Valencia (Valencia, Spain)

Paolo Boero

Università di Genova (Genova, Italy)

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To E. Fischbein, H. Freudenthal, R. Skemp,
and the other 81 colleagues who began this adventure
thirty years ago.

CONTENTS

RINA HERSHKOWITZ AND CHRIS BREEN

FOREWORD – EXPANSION AND DILEMMAS

There is nothing like a vision which turns into a reality. This volume was first conceived of in Rina's period of office as President in 2002 and has reached fruition during Chris's tenure in 2006 as we celebrate 30 years of PME. We both offer our sincere appreciation for all the cooperative work, insight and dedication that has been put into the project by the IC members who were the pioneers in processing the volume idea, the volume editors, authors and reviewers of the various chapters. Congratulations are due to all concerned!

This volume is intended to provide a window through which researchers in the mathematics education community as a whole, as well as researchers and students in other educational communities, can join with the insiders –PME members and their students– to gain an understanding of the scope and contribution of some of the various areas of PME research.

This volume also represents a milestone for the PME community as it celebrates 30 years of existence. The organization has grown considerably in numbers since its formation in 1976 and it is appropriate for PME to mark the occasion by presenting a consolidated report on important PME research interests, questions and methodologies, findings and conclusions, as well as some hints of possible future activity in the field.

Although neither of us had the opportunity to be one of the lucky few who were present at the start of PME, we are able to access some of the history through the introductory chapter of a previous PME milestone publication consisting of 7 chapters which was edited by Perla Nesher and Jeremy Kilpatrick in 1990. In the introductory chapter to that book the founder of PME and first PME President, Professor Efraim Fischbein, described the history of PME and how it came to be established. He then went on to analyze the similarities and differences between mathematics and psychology and why the two fields should communicate and cooperate for the sake of improving mathematics education (Fischbein, 1990).

A reading of both milestone PME books shows that we moved beyond debating this issue some time ago. It is clear that the learning and teaching of mathematics lies at the core of PME research, and that cognitive psychology generally forms the basis of the theoretical framework as well as most of the methodological tools, the data analyses as well as their interpretation and conclusions. In addition, this new PME30 volume showcases some new trends and tools, which interweave with the existing trends in the accumulation of the body of PME research work. We both believe that this ability to adjust to changing conditions and priorities clearly

A. Gutiérrez, P. Boero (eds.), Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future, ix–xii.

demonstrates the strength and openness of both the organization and its strong research focus.

In the last chapter of the 1990 research book, Nicolas Balacheff wrote on some "Future perspectives for research on psychology of mathematics education" (Balacheff, 1990, p. 138), and sketched each of them carefully and briefly. In this present PME30 research volume, it is noticeable that each of these named future perspectives has become a main topic of research and has been addressed in several chapters of the PME30 volume.

As we have said earlier, we believe that the way in which new trends and foci of attention have evolved and developed in parallel with well established fields over the years is one of the very positive characteristics of PME as a scientific group. We would like to highlight three examples of such trends.

Socio-cultural research trends, in its broadest meaning *(At least 3 explicit chapters in this volume)*

The start of this trend at PME was clearly announced in the middle of the eighties with the lecture of Bishop (1985), and this emphasis on the impact of socio-cultural influences, including socio and/or cultural influences on learning and teaching of mathematics has accelerated in the ensuing years as contextual influences on learning becoming better understood. Through this research the focus of attention has moved from the investigation of the subjects' constructing of mathematical knowledge in laboratory conditions to their own natural conditions. Some of this research has expanded beyond a focus on the immediate context of the group and classroom to include some of the broader influences of culture and society on the teaching and learning of mathematics.

Sub-trends of this socio-cultural research trend can be identified, such as:

– Classroom research. The focus has been on the very many interactions that the student has with other students in the classroom, with the teacher, with the mathematical tasks and with learning tools such as the computer.
– Research in which factors of equity and justice have become more pronounced.
– The effects of affective elements on learning and teaching mathematics.

This growing socio-cultural trend also touches on two other important trends, which have both becoming increasingly evident as a PME field of research in the past two decades:

Teaching, teachers and teacher education *(At least three explicit chapters in this volume)*

Research in this area has often been driven from a teacher-centered approach where the focus is on a need to know more about teaching styles, teacher beliefs, teacher decision-making, and different possibilities for improving mathematics teacher education. In this area, there are also an increasing number of reports of collaborative research undertaken by researchers with teachers as well as an interest in the way in which teachers might develop valid ways of researching their own practice. A second aspect of this research trend overlaps with classroom research where the teacher is seen as a partner of the learning/teaching process and interacts with students to influence the way in which they construct knowledge.

Computer tools in mathematics learning and teaching *(At least two explicit chapters in this volume)*

The source of this trend was influenced by the development and increased availability of computers and computerized tools as cultural artifacts. Research has focused on the introduction and potential contribution to the learning of mathematics of both "general" computerized tools such as Excel, as well as those, which were created especially for their use in learning mathematics. Researchers have been interested in the effects on the mathematics learning/teaching processes and on mathematics as a learning subject. Do computerized tools broaden or lessen the potential scope of the mathematical topic being investigated, or does this depend on the special context? The voice of this debate is still being heard. Another issue in this field that has been the focus of recent observation and analysis has been the processes by which students use computerized tools in learning mathematics in computerized environment, and the extent to which these tools can become assimilated as part of the students' inner instruments.

This brief summary of three recent trends in PME research highlights the way in which the field of research inevitably changes over a period of thirty years. We have previously commended the way in which PME as an organization has been able to naturally grow in size and expand its focus of attention over its thirty years of existence. As we celebrate the publication of this volume which commemorates these thirty years we would like to draw attention to the degree to which these three trends draw on different paradigms, theories and research methodologies from those which dominated past studies. As a research organization we have to consider the dilemmas that this raises for the community. We have to consider questions such as:

- What are the inner relationships between research paradigms and research methodologies? Do they fit together? (Schoenfeld, 1994). Do such methodologies exist? For example: Which methodologies are appropriate for a research domain such as classroom research?
- How do we ensure that we maintain an appropriate balance between the unifying trend of using established and familiar research methodological tools and applications of scientific language while, at the same time, embracing the insights that are emerging from new ideas and paradigms?
- How do we try to ensure that the unifying pressures of globalization of knowledge and internationally comparative research findings do not prevent us from ensuring that the critical voices of difference as regards research priorities, agenda and methodologies that have been silenced in a variety of ways across the world can still be heard at our annual conference?

We are both convinced that PME has an important and crucial role to play in addressing these questions. The strength of PME lies in the varied and textured nature of its research and in the way that it provides a dynamic forum for international mathematics educators to meet on an annual basis to exchange ideas and perspectives in a tolerant and open-minded conversational space. The annual meeting provides the seed for this international exchange of ideas through its various options of plenary presentations, research reports, research fora, and oral and poster presentations and particularly throught he Discussion Groups and Working Sessions. Many cutting edge developments in the field have resulted from these interactions and the way in which PME members have been prepared to engage with issues and dilemmas in constructive ways over the period of thirty years.

This volume provides a very welcome opportunity for us to take stock and acknowledge the rich and diverse spread of PME research and the significant contribution that this research makes to mathematics education across the world. We are both proud to be associated with its publication.

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Rina Hershkowitz (PME President 2001 - 2004[1])
Science Teaching Department
The Weizmann Institute of Science
Rehovot 76100 (Israel)
rina.hershkovitz@weizmann.ac.il

Chris Breen (PME President 2004 - 2007)
School of Education
University of Cape Town, Private Bag
Rondebosch, 7701 (South Africa)
cb@humanities.uct.ac.za;
http://www.chrisbreen.net

[1] Peter Gates was Acting President of PME for the last year of Rina's term of office.

ANGEL GUTIERREZ AND PAOLO BOERO

INTRODUCTION

The *International Group for the Psychology of Mathematics Education* (PME Group)[1] was founded in 1976 in Karlsruhe (Germany), during the ICME-3 Congress. The first meeting of the PME Group took place in Utrecht (The Netherlands) in 1977, organized by Hans Freudenthal. Since then, every year the PME Group has met somewhere in one of the five Continents. From the very beginning, the PME Group developed as one of the most interesting and successful examples of international co-operation in the field of educational research (not only in mathematics education!). Nowadays such development still continues –not exempt of tensions sometimes–, as a result of a continuous effort of the PME Group to evolve, to develop new ways of looking into mathematics teaching and learning, and to integrate new ideas coming from other scientific research fields. As a consequence, we believe, the PME Group is the most influential forum of research in mathematics education, and the proceedings of the PME conferences are a primary source of information for any researcher in mathematics education, since they summarize the state of the art at that time in the field (all the proceedings are available in ERIC; A list can be found at the end of this Introduction).

There are several reasons that can explain the success of that initiative: The human and scientific quality of the founding members; The fact that the growth of "PME" –as the Group is familiarly named– happened during the full development of mathematics education as a research domain, contributing to that development, but also profiting from it; The diversity of research traditions and developments in mathematics education, from all over the world, that converged and merged in the PME community, which motivated and fostered co-operation between different countries and research orientations; The intrinsic interest and fascination of a systematic co-operation between mathematics educators, psychologists and mathematicians engaged in the improvement of the teaching and learning of mathematics.

The aim of this volume is to account for the quantity and quality of the research work performed and enhanced within the PME Group since its origins, 30 years ago, with an eye to future developments. Such an enterprise was started by the PME International Committee, and designed according to the following steps:

– choice of the target public,

[1] Detailed information about the activities of the PME Group can be found in its web page http://igpme.org

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– choice of criteria for organizing the surveys,
– choice of areas for the different chapters, and
– choice of authors and reviewers.

This volume is willing to act as a point of reference for future research in mathematics education. It should be useful mainly to both researchers in mathematics education and mathematics teachers' trainers with any level of expertise in the field. However, particular attention was paid to the needs of people approaching the task of research in mathematics education –particularly PhD students– and to persons having difficulties in keeping regular contacts with research in mathematics education at the international level.

Having as a core aim of this volume to give an overview of the research made by the PME community, a decision was necessary respect to a question inherent to the scope of the synthesis to be made: The chapters could survey research contributions produced strictly within the PME Group (ignoring external developments or influences from the exterior), or could survey the whole field of research in mathematics education (within and out of the PME Group). An intermediate position was adopted, and the authors were requested to make an effort to focus mainly on the contributions presented at the PME conferences, but keeping also into account their possible origin out of the PME environment and their further developments in other media different from the proceedings of the PME conferences (articles on international journals, books, etc.).

This choice did not exclude relevant research orientations or research areas in mathematics education from the volume. Indeed, if we consider this volume and compare its contents with other survey volumes (handbooks) of research in mathematics education recently published, we must recognize that most research in mathematics education is well represented in the PME Group, and that many research contributions produced within the PME Group have been very influential in orienting research in several areas of mathematics education.

In concordance to the above mentioned core aim of this volume, the chapters were organized to provide a wide unbiased account of that research in specific areas. The structure of the chapters was planned to avoid two opposite styles of surveying: The "list style", resulting in a mere flat description of contributions, and the "unilateral reconstruction style", according to which an author interprets the evolution of research in a given area of mathematics education from his/her personal research interests and orientation. The guidelines given to the authors of the chapters reflected the decision to meet the challenge of writing personal surveys of a field that try to account for the variety of paradigms, positions and contributions apparent within the PME Group, to summarize key issues, to contrast or compare perspectives, to show the inner evolution of PME research in the given field, and also to include indications of possible potential developments for future research in the field. And this is what the authors have done!

The choice of areas for the different chapters was a delicate issue (even more delicate than the previous ones). Indeed it was not possible neither to consider only the present PME classification of research domains (which reflects the present status of our research), nor to create an abstract set of areas, trying to take into account "old" ones and "new" ones. The base for the choice of the areas of the different chapters was the present situation of the research in PME, but an effort was made through the choice of the authors and formal and informal contacts with them in order to ensure that they would keep into account both the present main orientations of research and their roots (this is evident particularly for those chapters dealing with areas that emerged as relevant areas in the last two decades, or even in the last decade).

On the other side, we have not tried to include in the volume a chapter for "every" research area present in the PME proceedings, but we have selected those research areas having a consolidated research agenda in the PME Group, more than just a set of unconnected reports. In some cases, the chapters evidence the characteristics of consolidated research areas, with research agendas smoothly evolving along the time. In other cases, the chapters well represent tensions that are inherent in the PME Group as a permanently "young" community, tensions inherent in the evolution of the cultural and scientific policy of the PME Group, and concerning the leading scientific orientations for the future.

The choice of authors was not intended to "give a prize" to someone, or to recognize a leading personality in a given field of research. Besides a high scientific quality and great expertise in the area of a chapter, the facts of being within the PME research community for a long time, having had responsibilities in promoting working groups, discussion groups, or having been a member of the International Committee, were considered as qualities that could ensure the authors would have necessaries knowledge of the PME activity, experience, capability, and equilibrium and distance from personal research orientations to ensure that they would write top quality chapters. With them, some young but knowledgeable researchers have collaborated in the writing of some chapters.

On the other side, an effort was also made to represent the variety of regions and cultural orientations present within the PME Group. Here we must recognize that it was not easy to get a final group of authors as representative as we intended, mainly due to the barrier of native languages –a delicate issue in an international organization like the PME Group. This caused the set of authors to be more biased in favour of the English-speaking countries than the present distribution of PME people is[2].

A strategy we could use for some chapters was to form teams of researchers from different countries and research traditions (or scientific orientations) to cooperate in writing the same chapter. We must say that we were very lucky in having joined those groups of authors; It would have been very difficult to make better teams.

As concerns the reviewing process, it was decided that each chapter had to be reviewed by two PME members, both having great expertise in the area of the paper, one of them near to the scientific orientation of the authors, and the other far from it. A main difficulty was to get people who, fitting the scientific requirements,

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2 60% of the authors, against an average of almost 44% of PME members in 2000-2005, are native English speakers. On the other side, 37% of the reviewers are native English speakers.

could accept the difficult task of reviewing a survey chapter, in some cases in a narrow time.

The volume includes 15 chapters, organized into five sections, devoted to the main research domains of interest to the PME Group. The first section summarizes cognitively oriented research on learning and teaching specific mathematics content areas –algebra, geometry and measurement, and numbers. The second section has the same orientation as the first one, but it focuses on mathematical transversal areas –young children's development, advanced mathematical thinking, deductive reasoning, and visualization.

The third section presents the PME research centred on the teaching and learning of mathematics in technologically rich environments; Most research on computer based environments by the PME Group was centred on algebra, calculus, and geometry, so the chapters of this section summarize the research on those mathematics contents.

The fourth section is devoted to the research on social aspects of mathematics education like affect, gender, equity, constructivism, and other socio-cultural elements influencing the teaching and learning of mathematics.

Finally, the fifth section includes two chapters summarizing the PME research on teachers training and professional life of mathematics teachers.

Coming to the end of this Introduction, we want to acknowledge the enormous effort of the authors, who unselfishly have devoted a great part of their time to write the top quality chapters in this volume, and the reviewers, who have also devoted many time to make constructive comments on the chapters to raise still more their quality -the names of the reviewers can be found at the end of this Introduction. We must also acknowledge that the task of co-ordinating the production of this volume was made easier by some circumstances: The commitment of the International Committee, particularly our President, in giving us advice in several crucial moments of the production; The collaborative style of work of authors and reviewers, who have made their best to ensure the quality of the chapters; The help of the PME executive secretary and other PME members, who provided the authors or the editors with first hand information about the early years of the PME Group.

Lastly, we want to use these pages to render a tribute to all people who have collaborated in making so successful the project, initiated 30 years ago, of creating a place, the PME Group, where any person concerned with the mathematics education could meet other persons with similar interests and share their research activity:

– Thanks to the thousands of (more or less) anonymous mathematics researchers who have felt the need to communicate with their colleagues and have found in the PME meetings an adequate place. The PME Group is alive thanks to all them.
– Thanks to those persons who volunteered as members of the International Committee. They assumed the responsibility of driving the PME Group and taking decisions affecting its scientific orientation. In particular, special thanks to the Presidents of the PME Group.

– Thanks to Joop van Dormolen, for many years the executive secretary of the PME Group. He has been a tireless worker in the shadow, taking care of every administrative aspect of the Group and making easier others' work.
– Thanks to all those anonymous persons, from undergraduate students to senior researchers, who, year after year, accepted the responsibility of collaborating in the organization of a PME conference. Those having had such experience know that, for a year, their main endeavour was to work to have everything ready to ensure the success of the next conference.

Angel Gutiérrez
Departamento de Didáctica de la Matemática
Universidad de Valencia
Apartado 22045
46071 Valencia (Spain)
angel.gutierrez@uv.es
http://www.uv.es/Angel.Gutierrez

Paolo Boero
Dipartimento di Matematica
Università di Genova
Via Dodecaneso, 35
16146 Genova (Italy)
boero@dima.unige.it

REVIEWERS OF THE CHAPTERS IN THIS VOLUME

The following persons collaborated with the editors of this volume by reviewing one of its chapters:

Janet Ainley (UK)
Abraham Arcavi (Israel)
Nicolas Balacheff (France)
Mariolina Bartolini Bussi (Italy)
José Carrillo (Spain)
Paul Cobb (USA)
Martin Downs (Greece)
Jeff Evans (UK)
Athanasios Gagatsis (Greece)
Nuria Gorgorió (Spain)
Markku Hannula (Finland)
Lulu Healy (Brazil)
Barbara Jaworski (UK)
Keith Jones (UK)
Jean-Baptiste Lagrange (France)
Carolyn Maher (USA)
Nicolina Malara (Italy)
Joanna Mamona-Downs (Greece)
John Monaghan (UK)
Pearla Nesher (Israel)
Bernard Parzysz (France)
Erkki Pehkonen (Finland)
Demetra Pitta (Cyprus)
David Reid (Canada)
Adalira Sáenz-Ludlow (Colombia/USA)
Judith Sowder (USA)
David Tall (UK)

PRESIDENTS OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

The following persons acted as Presidents of the PME Group:

Efraim Fischbein (Israel)	1977 - 1980
Richard R. Skemp (UK)	1980 - 1982
Gérard Vergnaud (France)	1982 - 1984
Kevin F. Collis (Australia)	1984 - 1986
Pearla Nesher (Israel)	1986 - 1988
Nicolas Balacheff (France)	1988 - 1990
Kathleen M. Hart (UK)	1990 - 1992
Carolyn Kieran (Canada)	1992 - 1995
Stephen Lerman (UK)	1995 - 1998
Gilah C. Leder (Australia)	1998 - 2001
Rina Hershkowitz (Israel)	2001 - 2004
Chris Breen (South Africa)	2004 - 2007

LOCAL ORGANIZERS OF THE PME INTERNATIONAL CONFERENCES

The following persons acted as Chairs of the local organizing committees of the PME International Conferences:

No.	Year	Chair	Place
PME 1	1977	Hans Freudenthal	Utrecht (Holland)
PME 2	1978	Elmar Cohors-Fresenborg	Osnabrück (Germany)
PME 3	1979	David Tall	Warwick (UK)
PME 4	1980	Robert Karplus	Berkeley (USA)
PME 5	1981	Claude Comiti	Grenoble (France)
PME 6	1982	Alfred Vermandel	Antwerpen (Belgium)
PME 7	1983	Rina Hershkowitz	Shoresh (Israel)
PME 8	1984	Beth Southwell	Sidney (Australia)
PME 9	1985	Leen Streefland	Noordwijkerhout (Holland)
PME 10	1986	Leone Burton and Celia Hoyles	London (UK)
PME 11	1987	Jacques C. Bergeron	Montreal (Canada)
PME 12	1988	Andrea Borbás	Veszprem (Hungary)
PME 13	1989	Gérard Vergnaud	Paris (France)
PME 14	1990	Teresa Navarro de Mendicuti	Oaxtepex (Mexico)
PME 15	1991	Paolo Boero	Assisi (Italy)
PME 16	1992	William E. Geeslin	Durham (USA)
PME 17	1993	Nobuhiko Nohda	Tsukuba (Japan)
PME 18	1994	João Pedro da Ponte	Lisbon (Portugal)
PME 19	1995	Luciano Meira	Recife (Brazil)
PME 20	1996	Angel Gutiérrez	Valencia (Spain)
PME 21	1997	Erkki Pehkonen	Lahti (Finland)
PME 22	1998	Alwyn Olivier	Stellenbosch (South Africa)
PME 23	1999	Orit Zaslavsky	Haifa (Israel)
PME 24	2000	Tadao Nakahara	Hiroshima (Japan)
PME 25	2001	Marja van den Heuvel-Panhuizen	Utrecht (Holland)
PME 26	2002	Anne D. Cockburn	Norwich (UK)
PME 27	2003	A. J. (Sandy) Dawson	Honolulu (USA)
PME 28	2004	Marit J. Høines	Bergen (Norway)
PME 29	2005	Helen L. Chick	Melbourne (Australia)
PME 30	2006	Jarmila Novotná	Prague (Czech Republic)

PROCEEDINGS OF THE PME CONFERENCES

The table below indicates the ERIC ED numbers for the proceedings of the PME International Conferences. Many PME proceedings can also be freely retrieved from the ERIC web page http://www.eric.ed.gov/.

PME INTERNATIONAL GROUP CONFERENCES

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	Not available
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, The Netherlands	ED411130 (v. 1), ED411131 (v. 2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128 (v. 1), ED411129 (v. 2)
13	1989	Paris, France	ED411140 (v. 1), ED411141 (v. 2), ED411142 (v. 3)
14	1990	Oaxtepex, Mexico	ED411137 (v. 1), ED411138 (v. 2), ED411139 (v. 3)
15	1991	Assisi, Italy	ED413162 (v. 1), ED413l63 (v. 2), ED413164 (v. 3)
16	1992	Durham, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (v. 1), ED411135 (v. 2), ED411136 (v. 3)
20	1996	Valencia, Spain	ED453070 (v. 1), ED45307 1 (v. 2), ED453072 (v. 3), ED453073 (v. 4), ED453074 (addenda)
21	1997	Lahti, Finland	ED416082 (v. 1), ED416083 (v. 2), ED4l6084 (v. 3), ED416085 (v. 4)
22	1998	Stellenbosch, South Africa	ED427969 (v. 1), ED427970 (v. 2), ED427971 (v. 3), ED427972 (v. 4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301 (v. 1), ED452302 (v. 2), ED452303 (v. 3), ED452304 (v. 4)
25	2001	Utrecht, The Netherlands	ED466950
26	2002	Norwich, United Kingdom	ED476065

No.	Year	Place	ERIC number
27	2003	Hawai'i, USA	Being processed. Also available in http://onlinedb.terc.edu.
28	2004	Bergen, Norway	Being processed.
29	2005	Melbourne, Australia	Being processed.

PME NORTH AMERICAN CHAPTER (PME-NA) CONFERENCES

The table below indicates the ERIC ED numbers for the proceedings of PME-NA Conferences published separately from the procedings of PME International Conferences. Many PME-NA proceedings can also be freely retrieved from the ERIC web page http://www.eric.ed.gov/.

No.	Year	Place	ERIC number
3	1981	Minnesota, USA	ED223449
4	1982	Georgia, USA	ED226957
5	1983	Montreal, Canada	ED289688
6	1984	Wisconsin, USA	ED253432
7	1985	Ohio, USA	ED411127
8	1986	Michigan, USA	ED301443
10	1988	Illinois, USA	ED411126
11	1989	New Jersey, USA	ED411132 (v. 1), ED411133 (v. 2)
13	1991	Virginia, USA	ED352274
15	1993	Califomia, USA	ED372917
16	1994	Louisiana, USA	ED383533 (v. 1), ED383534 (v. 2)
17	1995	Ohio, USA	ED389534
18	1996	Florida, USA	ED400178
19	1997	Illinois, USA	ED420494 (v. 1), ED420495 (v. 2)
20	1998	North Carolina, USA	ED430775 (v. 1), ED430776 (v. 2)
21	1999	Cuernavaca, Mexico	ED433998
22	2000	Arizona, USA	ED446945
23	2001	Utah, USA	ED476613
24	2002	Georgia, USA	ED471747
26	2004	Toronto, Canada	Being processed.
27	2005	Virginia, USA	Being processed.

CAROLYN KIERAN

RESEARCH ON THE LEARNING AND TEACHING OF ALGEBRA

A Broadening of Sources of Meaning

The learning and teaching of algebra has always been a fundamental and vibrant stream of the research carried out within the PME community – from the 1st PME conference in 1977 to the 29th in 2005 when 33 algebra research reports were presented. While the themes of the past are still to be found in the algebra research of the present, there have been some major shifts. Earlier research tended to focus on algebraic concepts and procedures, algebra word problem solving, and students' difficulties in making the transition from arithmetic to algebra. The letter-symbolic was the primary algebraic mode that was investigated and theoretical frameworks for analyzing research data rarely went beyond the Piagetian. However, over time, PME algebra research broadened to encompass other representations, the use of technological tools, different perspectives on the content of algebra, and a wide variety of theoretical frameworks for thinking about algebra learning/teaching and for analyzing data. Each new phase of the research, in fact, embedded ongoing themes as it expanded its scope in an attempt to more fully explain algebra learning and teaching. This process, the highlights of which are the subject of this chapter, is one of ever-increasing complexity as researchers continue to develop theoretical and empirical foundations for their descriptions of algebra teaching and for the meaning making, and its myriad paths, engaged in by students of algebra.

Examples have been drawn from the body of algebra research over the 30-year history of PME. They reflect the main themes of interest of PME algebra researchers over the years. Some date back to the first years and continue to today; others are more recent. In order to capture the newer interests as they came into existence, the main highlights are presented more or less chronologically according to themes – in fact, three theme-groups, which emerged over the 30-year history of PME algebra research (see Table 1). The first theme-group, which points to the interests of PME algebra researchers from the start of PME in 1977, includes a focus on the transition from arithmetic to algebra, on variables and unknowns, equations and equation solving, and algebra word problems. During approximately the mid-1980s, the field of PME algebra research witnessed the growth of themes that reflected an interest in algebra as generalization, and a focus on multiple representations and the use of new technological learning tools. Then, during the mid-1990s, PME algebra research began to encompass themes related to algebraic

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thinking among elementary school students, a focus on the algebra teacher and algebra teaching, as well as on the learning of algebra within dynamic environments that included the dynamic modeling of physical situations.

Table 1. Major themes that have emerged over the 30-year history of PME algebra research from 1977 to 2006.

Time Period	*Theme-Groups that emerged*
1977 to 2006	1. Transition from arithmetic to algebra, variables and unknowns, equations and equation solving, and algebra word problems.
Mid-1980s to 2006	2. Use of technological tools and a focus on multiple representations and generalization.
Mid-1990s to 2006	3. Algebraic thinking among elementary school students, a focus on the algebra teacher/teaching, and dynamic modeling of physical situations and other dynamic algebra environments.

However, the new themes that arose during these time periods did not simply "join" the existing research. Changing theoretical and methodological frameworks over the years altered the ways in which even the earliest themes came to be researched. Thus, the story to be recounted in this chapter, while it deals primarily with the main theme-groups of the research conducted by members of the PME community, also attempts to suggest the ways in which the process of conducting this research changed as theoretical and methodological frames shifted. The first three sections of this chapter treat the three main theme-groups with their accompanying theoretical and methodological considerations. The fourth and final section of the chapter integrates these themes within a discussion of sources of meaning in algebra learning, and demonstrates that research on the ways in which students derive meaning in algebra has widened considerably in perspective over the years.

The body of literature that forms the foundation for this review chapter is obviously the PME Proceedings during the past 30 years. However, when a PME research report was further elaborated and subsequently published in a journal article or book, it is the journal article or book that will generally be cited. As well, for PME research that has been situated within a broader field of research, a reference to this broader field will often be provided. It is also noted that additional PME algebra resources exist –resources that more fully discuss some of the ideas referred to in this chapter– for example, the chapter, "Cognitive Processes Involved in Learning School Algebra," in the PME monograph, *Mathematics and Cognition* (Nesher & Kilpatrick, 1990), and the volume, *Perspectives on School Algebra* (Sutherland, Rojano, Bell & Lins, 2001), which was the culmination of the 1990s activity of the PME Working Group on Algebraic Processes and Structure. Other sources in the literature to which members of the PME community of algebra researchers have contributed include: Wagner & Kieran (1989), Grouws (1992), Bednarz, Kieran & Lee (1996), Bishop, Clements, Keitel, Kilpatrick & Laborde (1996), Chick, Stacey, Vincent & Vincent (2001), Kieran, Forman & Sfard (2001), English (2002), Mason & Sutherland (2002) and Stacey, Chick & Kendal (2004).

THEME-GROUP ONE: THE ORIGINAL STRANDS WITH THEIR FOCUS ON THE TRANSITION FROM ARITHMETIC TO ALGEBRA, ON VARIABLES AND UNKNOWNS, EQUATIONS AND EQUATION SOLVING, AND ALGEBRA WORD PROBLEMS

In its early years, the PME community of algebra researchers generally considered the algebra curriculum as a given, focusing attention primarily on students' thinking and methods as they encountered the symbols and procedures that were the standard algebra fare in the beginning years of high school (Kieran, 1979, 1990). The historical development of algebra as a symbol system served both as a backdrop for researchers' thinking about the evolution in students' usage of symbolism (see, e.g., Harper, 1981, 1987) as well as a theoretical framework for more broadly interpreting empirical data of algebra learning research (e.g., Filloy, 1990). An additional perspective that was current in those early years was the view of algebra as generalized arithmetic (Booth, 1981, 1984). Meaning for algebra was to be derived from its numerical foundations, but as well it was expected that students would learn to see structural similarities and equivalences in the expressions and equations they transformed. However, the fact that the signs and symbols in algebra were to be interpreted differently from the ways in which they were interpreted in arithmetic created discontinuities for students beginning algebra.

Interpreting algebraic signs, unknowns, and variables

While arithmetic and algebra share many of the same signs and symbols, such as the equal sign, addition and subtraction signs, even the use of letters (e.g., Area (A) = Length (L) x Width (W)), many conceptual adjustments are required of the beginning algebra student as these signs and symbols shift in meaning from those commonly held in arithmetic. Early research on the ways in which students interpret algebraic symbols tended to focus on cognitive levels (Küchemann, 1981), prior arithmetic experience and methods of thinking (Booth, 1984), and difficulty with notation such as the equal sign with its multiple meanings (Kieran, 1981; Vergnaud, 1984) and the use of brackets (Kieran, 1979). Included in this strand of research were teaching experiments aimed at, for example, extending students' meaning for the equal sign from the "do-something signal" (Behr, Erlwanger & Nichols, 1976) prevalent in elementary school arithmetic reasoning to a consideration of the symmetric and transitive character of equality (Vergnaud, 1988) and its use in algebraic equations (e.g., Herscovics & Kieran, 1980).

In coming to make sense of algebra, students' prior arithmetical usage of letters in formulas and as labels has to make room for letters as unknowns and as variables, and later as parameters. Early research on students' thinking about the literal symbols of algebra disclosed a host of undeveloped interpretations (see Collis, 1974; Küchemann, 1981; Wagner, 1981; Clement, 1982). In a teaching experiment designed specifically to encourage the acquisition of the notion of letter as generalized number, Booth (1982, 1983) found strong resistance on the part of students to the assimilation of this idea. Among the large number of PME studies that have continued to be devoted to researching students' difficulties in understanding various aspects of the concept of variable is the work of Ursini and her colleagues (e.g., Ursini, 1990; Ursini & Trigueros, 1997; Trigueros & Ursini, 1999), Fujii (1993) and Bills (1997, 2001). Also included in this body of work is the research of Stacey and MacGregor (1997) on the presence of multiple referents and shifts in the meaning of the unknown, and that of Vlassis (2004) on the polysemy of the minus sign in algebraic expressions and equations. In studies involving older students, Furinghetti and Paola (1994) have found that only a small minority could adequately describe differences among parameters, unknowns, and variables (see also Bloedy-Vinner, 1994, 2001, on parameters, and Kopelman, 2002, on free and bound variables).

Some of the recent work within this theme group has focused on additional factors that impinge on students' interpretation of algebraic notation: For example, what one is able to perceive and prepared to notice (Sfard & Linchevski, 1994) and the nature of the instructional activity (Wilson, Ainley & Bills, 2003). These latter studies suggest some of the changes in perspective present in current research that nevertheless continues the study of a conceptual area that has been of research interest since PME's first days.

Working with expressions, equations, and equation solving

Early PME research involving algebraic forms disclosed students' difficulties in interpreting expressions such as $a + b$ as both *process* and *name/object* (an obstacle related to the findings of Davis, 1975, and to the notion of *acceptance of lack of closure*, developed by Collis, 1974). Teaching experiments intended to help students construct meaning for expressions by means of, for example, rectangular area models (Chalouh & Herscovics, 1988) and for equations by means of arithmetic identities (Herscovics & Kieran, 1980) suggested that students could more easily construct meaning for equations than for expressions. This was borne out by the results of the research of Wagner, Rachlin and Jensen (1984) who noted that students tried to add "= 0" to any expression they were asked to simplify.

A significant number of early studies focused on the equation-solving procedures of beginning algebra students: (i) Intuitive approaches, which included the use of number facts, counting techniques, and cover-up methods (Bell, O'Brien & Shiu, 1980; Booth, 1983); (ii) Trial-and-error substitution (Kieran, 1985); and (iii) Formal methods (Whitman, 1976; Kieran, 1983, 1988). Equation-solving errors of students have also been of research interest: For example, ignoring the minus sign preceding a pair of numbers to be combined (Herscovics & Linchevski, 1994); reduction errors (Carry, Lewis & Bernard, 1980); and erroneous checking behavior (Perrenet & Wolters, 1994; Pawley, 1999). More recent research has also begun to inquire into the methods used by students to solve inequalities (e.g., Tsamir & Bazzini, 2002), noting that students intuitively use the balance model to do the same thing to both sides of an inequality, and continue to draw on equation-related analogies when dividing both sides by a not-necessarily positive value. Systems of equations have, of late, also attracted research attention. Filloy, Rojano and Solares (2003, 2004), who observed students working with problems that can be solved by systems of equations, found that these students, who had already been introduced to the solving of one-unknown linear equations, tended to make more sense of comparison-based than substitution-based solving methods.

Solving algebra word problems

Another of the discontinuities faced by beginning algebra students is the introduction of formal representations and methods to solve problems that, up to this stage in their schooling, have been handled intuitively. Because arithmetic is largely procedural, students are used to thinking about the operations they use to solve a problem rather than the operations that they should use to represent the relations of the problem situation. Not only does setting up an equation require a different way of thinking about the problem, the beginning algebra student must also learn to solve the equation with procedures that yield successive equivalent equations until the solution is found. Filloy and Rojano (1989) have pointed out that a *didactical cut* occurs between equations of the type $ax + b = c$ that can be solved by arithmetic methods, and equations of the type $ax + b = cx + d$ which necessitate formal algebraic methods. They explored the use of various concrete models (both the balance and geometric area models) in the teaching of formal equation-solving methods and found that these models did not significantly increase most students' abilities to operate formally with equations having two occurrences of the unknown.

Additional research on the use of concrete models and manipulatives in equation solving includes a study by Boulton-Lewis, Cooper, Atweh, Pillay, Wilss and Mutch (1997). They reported that the 8th graders (approximately 14 years of age) of their study did not use the concrete manipulatives (cups, counters, and sticks) that were made available for solving linear equations, and suggested that the concrete representations increased processing load. In contrast, advocates of such models (e.g., Linchevski & Williams, 1996; Radford & Grenier, 1996) argue that the balance scale facilitates the understanding of the operation of eliminating the same term from both sides of an equation. In particular, Vlassis (2002) found that, for the two classes of 8th graders whom she observed over a period of 16 lessons, the balance model was an effective tool in conveying the principles of transformation. However, according to Vlassis, if students have not already extended their numerical range to encompass the negative integers, cancellation errors will be inevitable (for further research on negative numbers, see Gallardo, 2002).

Continuing research on representing the relationships within typical word problems has produced evidence of students' preference for arithmetic reasoning and difficulties with the use of equations to solve word problems (e.g., Bednarz & Janvier, 1996; Cortés, 1998). Stacey and MacGregor (1999) have observed that, at every stage of the process of solving problems by algebra, students were deflected from the algebraic path by reverting to thinking grounded in arithmetic problem-solving methods. Another study (Van Amerom, 2003) dealing with the setting up of equations within the activity of word-problem-solving has reported that, for 6th

and 7th graders (about 12 to 13 years of age), reasoning and symbolizing appear to develop as independent capabilities; that is, students could write equations to represent problems, but did not use these equations to find the solution, preferring instead to use more informal methods. However, as is suggested by the examples of student work that are presented by Van Amerom, the students did use the equations they had generated as a basis for their reasoning about the problems and for finding their solutions. Similarly, Malara (1999) has observed students successfully using numerical substitution within problem representations in order to arrive at solutions. Kutscher and Linchevski (1997) have also noted the beneficial effect of numerical instantiations as mediators in solving word problems. De Bock, Verschaffel, Janssens and Claes (2000), who have explored the influence of authentic and realistic contexts and of self-made drawings on the illusion of linearity in length and area problems, found no beneficial effect of the authenticity factor, nor of the drawing activity, on pupils' performance. Furthermore, the researchers suggest that realistic problems may, in fact, steer pupils away from the underlying mathematical structure of a problem.

Noticing structure

Some of the early research on equation solving also addressed the issue of students' awareness of the structure of algebraic equations and expressions (e.g., Kieran, 1989). For instance, Wagner, Rachlin and Jensen (1984) found that algebra students have difficulty dealing with multi-term expressions as a single unit and do not perceive that the structure of, for example, $4(2r+1) + 7 = 35$ is the same as $4x + 7 = 35$. According to Kieran (1984), students also find it demanding to judge, without actually solving, whether equations such as $x + 37 = 150$ and $x + 37 – 10 = 150 + 10$ are equivalent, that is, whether they have the same solution. More recently, Linchevski and Livneh (1999) found that 12-year-old students' difficulties with interpreting equations containing several numerical terms and an unknown were a reflection of the same difficulties that they experienced in purely numerical contexts. The researchers suggested that algebra instruction be designed to foster the development of structure sense by providing experience with equivalent structures of expressions and with their decomposition and recomposition. In a study of 11th graders (approximately 17 years of age), Hoch and Dreyfus (2004) found that very few students used structure sense and those who did so were not consistent. The presence of brackets seemed to help students see structure, focusing their attention on like terms and breaking up long strings of symbols. However, as noted by Demby (1997), students are poor at identifying structure, in particular, the properties they use when they transform algebraic expressions. As well, Hoch and Dreyfus (2005) have reported the difficulties that students experience with seeing that expressions such as $(x-3)^4 – (x+3)^4$ can be treated as a difference of squares.

The kinds of errors that students tend to make in algebraic manipulative activity have suggested to some researchers (e.g., Kirshner, 1989) that it is not an absence of theoretical/structural control that is the issue but rather a misperception of form.

In a study that extended Kirshner's earlier work on the visual syntax of algebra, Kirshner and Awtry (2004) reported on research aimed at investigating the role of *visual salience*[1] in the initial learning of algebra. They found that students did indeed engage with the visual characteristics of the symbol system in their initial learning of algebra rules: The percentage-correct scores for recognition tasks were significantly higher for visually salient rules than for non-visually-salient rules. Similarly, Hewitt (2003), in a study of 40 teachers and a class of 11- to 12-year-olds, found that the inherent mathematical structure, and the visual impact of the notation itself, had an effect on the way in which an equation was manipulated. (Before concluding this section, it is noted that a review of the PME studies on the learning of abstract algebra –e.g., Leron, Hazzan & Zazkis, 1994; Chin & Tall, 2000; Iannone & Nardi, 2002– and of linear algebra –e.g., Rogalski, 1996; Sierpinska, Trgalová, Hillel & Dreyfus, 1999; Maracci, 2003– is beyond the scope of this chapter.)

Additional remarks

The research reviewed in this first section has focused primarily on the themes initially seen during the early years of PME; nevertheless, as has been seen, research on those strands has continued to the present day. However, more recent work in this theme-group has reflected the theoretical and methodological shifts that have been occurring over the last two and a half decades. The theoretical framework of constructivism blossomed in the 1980s (see the plenary papers by Kilpatrick, Sinclair, Vergnaud & Wheeler in the 1987 PME Proceedings), attracting PME researchers with its notion that knowledge is actively constructed by the cognizing subject, not passively received from the environment (in this regard, see also Cobb & Steffe, 1983; Confrey, 1987). This theoretical perspective led algebra researchers to move their attention from, for example, the errors made by students to the ways in which they craft their understandings of algebraic concepts and procedures. While the focus was still cognitive, it tended to be much broader than that suggested by the research analyses carried out during PME's first few years.

During the 1990s, an additional change occurred. The socio-cultural theoretical perspective, which had developed outside the PME community, began to emerge within PME (see John-Steiner's, 1995, plenary paper presented at the PME19 conference in Brazil). The earlier constructivist/cognitive orientation shifted, for a large number of PME algebra researchers, toward analyses of social factors affecting algebra learning, with an accompanying interest in the mediating role of cultural tools (see, e.g., Meira, 1995, 1998; see also Cobb, 1994; Cobb & Yackel, 1995). This shift also led to an increase in classroom-based studies with a focus on teacher-student and student-student discourse (e.g., Bartolini Bussi, 1995; Sfard, 2001).

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[1] According to the researchers, "visually salient rules have a visual coherence that makes the left- and right-hand sides of the equations appear naturally related to one another" (p. 229). While $(x^y)^z = x^{yz}$ is considered a visually-salient rule, $x^2 - y^2 = (x - y)(x + y)$ is not.

The impact of the advent of technology in the mid-1980s, in combination with first one and then the other of the above two theoretical perspectives, will be noted in the next section, where the use of technological tools in algebra learning will be one of the principal foci of discussion.

THEME-GROUP TWO: THE USE OF TECHNOLOGICAL TOOLS IN ALGEBRA LEARNING AND A FOCUS ON MULTIPLE REPRESENTATIONS AND GENERALIZATION

During the 1980s, the primarily letter-symbolic emphasis of algebra research began to experience some movement. While the study of functions had been considered a separate domain of study in the years prior, the two began to merge in algebra research. Functions, with their graphical, tabular, and symbolic representations, came to be seen as *bona fide* algebraic objects (Schwartz & Yerushalmy, 1992). Graphical representations, in particular, began to be viewed as tools for infusing letter-symbolic representations with meaning (Romberg, Fennema & Carpenter, 1993). Some mathematicians and mathematics educators (e.g., Fey, 1984) argued that computing technology could have significant effects on the content and emphases of school-level and university-level mathematics. Early visionaries promoted the idea that computing technology could be harnessed to more fully integrate the multiple representations of mathematical objects in mathematics teaching. Soon a functional perspective could be seen throughout algebraic activity, but especially in the solution of “real world” problems by methods other than by-hand symbol manipulation, such as technology-supported methods (Fey & Good, 1985). This functional perspective has been summarized by Heid (1996) as follows: “The functional approach to the emergence of algebraic thinking ... suggests a study of algebra that centers on developing experiences with functions and families of functions through encounters with real world situations whose quantitative relationships can be described by those models” (p. 239).

Research on functions and multiple representations, without technological tools

Even before the use of technological tools to support the learning of multiple representations began to be studied by researchers, the difficulties experienced by students with graphical representations had been a theme of interest at PME (e.g., Kerslake, 1977; Janvier, 1981; Clement, 1985; Ponte, 1985). Kerslake (1977), for example, showed that in reading graphs representing the time taken to traverse mountain roads of various steepness, students confused the graph with the shape of the road itself. In a study of students’ work with diverse representations, Dreyfus and Eisenberg (1981) found that high-ability students preferred graphical representations, while low-ability students preferred the tabular.

For many researchers, understanding the connections between equations and graphs is considered fundamental to developing meaning for various algebraic representations and continues to be central to PME research. Drawing on studies by Moschkovich, Schoenfeld and Arcavi (1993), Knuth (2000) examined 9^{th} to 12^{th}

grade students' understanding of the concept that the coordinates of any point on a line will satisfy the equation of the line, within the context of problems that require the use of this knowledge. Knuth found an overwhelming reliance on letter-symbolic representations, even on tasks for which a graphical representation seemed more appropriate. The findings also indicated that for familiar routine problems many students had mastered the connections between the letter-symbolic and graphical representations; however, such mastery appeared to be superficial at best.

Zaslavsky, Sela and Leron (2002) have found evidence of much confusion regarding the connections between algebraic and geometric aspects of slope, scale and angle. Responses to a simple but non-standard task concerning the behavior of slope under a non-homogeneous change of scale revealed two main approaches – analytic and visual– as well as combinations of the two. The researchers recommended that instruction on slope distinguish between the erroneous conception of 'visual slope' –the slope of a line (for which the angle is a relevant feature)– and the 'analytic slope' –the rate of change of a function.

Algebra as generalization activity

"Algebra as generalization" is a perspective that has its PME roots in the use of algebraic notation as a tool for expressing proofs (e.g., Bell, 1976; Fischbein & Kedem, 1982; Mason & Pimm, 1984). The position that generalization is also a route to algebra was developed by Mason, Graham, Pimm and Gowar (1985; see also Mason, Graham & Johnston-Wilder, 2005). In some of the pioneering PME research on the use of algebraic notation as a tool for expressing general and figural patterns, and for justifying equivalent forms of these patterning relations, Lee (1987) and Lee & Wheeler (1987) found that few students use algebra or appreciate its role in justifying a general statement about numbers. Similar findings have been reported by MacGregor and Stacey (1993), who observed that an additional difficulty lies in students' inability to articulate clearly the structure of a pattern or relationship using ordinary language.

Healy and Hoyles (1999) have pointed out that the visual approaches generated in tasks involving generalization of matchstick patterns can provide strong support for the algebraic representation of sequences and the development of a conceptual framework for functions, but emphasized that there is a need to work hard to connect the observed number patterns to symbolic form. Ainley, Wilson and Bills (2003), in a report that compared *generalization of the context* with *generalization of the calculation*, found that generalizing the context did not seem to be sufficient to support pupils in moving to a symbolic version of the rule. However, Radford (2000), who has studied the transition from the particular to the general, has argued that such processes take time.

In a study that focused on the role of tabular representations, Sasman, Olivier and Linchevski (1999) presented 8^{th} graders (14-year-olds) with generalization activities in which they varied the representation along several dimensions, namely the type of function, the nature of the numbers, the format of tables, and the structure of pictures. Results showed that varying these dimensions had little effect on students' thinking. According to Mason (1996), a scrutiny of established school practice involving generalizing in algebra reveals that often, starting from geometric figures or numeric sequences, the emphasis is on the construction of tables of values from which a closed-form formula is extracted and checked with one or two examples. This approach in effect short-circuits all the richness of the process of generalization. In fact, it has been suggested by other researchers (e.g., Moss, 2005) that tabular representations may actually get in the way of students' coming to see the general relationships underlying patterns and representing them algebraically. Mason suggests several possible investigative approaches that can lead to students' construction of algebraic formulas, including visualization and manipulation of the figure on which the generalizing process is based.

Related to the research on generalization and justification is the body of PME work on elementary number-theoretic activity. For example, Garuti, Boero and Lemut (1998), in a study involving problems of the type, "Prove that the sum of two consecutive odd numbers is a multiple of 4," found that students needed to gradually learn how to explore and transform the given statement in order to construct a proof. Additional difficulties centered on students' not seeing how to connect their proving work to algebraic notation. Zazkis and Campbell (1996), who studied participants' understanding of the property that "every n^{th} number is divisible by n" found that such properties are not among those that the majority of students are familiar with. Some strategies for helping students become aware of such number-theoretic properties have been described by Mason (2002) and by Guzmán and Kieran (2002). (See also the report of the 2002 PME Research Forum on number theory.)

In research involving older students, Arzarello, Bazzini and Chiappini (1994) observed a link between expressing the relations of number-theoretic problems in suitable algebraic code and success in determining the solution to such problems. According to the researchers, appropriate naming is linked to the anticipatory aspects of solving and allows students to orient the solution process toward the aims of the problem. They also pointed out that some students, who could express the elements of a problem by using natural language, were unable to express them using the algebraic language. In contrast, Douek (1999) found that non-standard representations of number-theoretic problems did not prevent students from producing valid proofs and problem solutions. However, Alcock and Weber (2005) have reported, based on their research involving proof-related tasks, that students use both referential (involving some instantiation) and syntactic approaches (involving manipulation); and that students who use referential approaches may have a more meaningful understanding of their proofs but may not be able to complete them.

Early research on the use of technological tools in algebra learning

While the growth in interest in multiple representations was fed by the advent of technology, some of the early PME algebra research involving technology focused rather on programming. For example, a study by Soloway, Lochhead and Clement (1982) showed that students could better cope with translating a word problem into an "equation" when the equation was in the form of a short computer program specifying how to compute the value of one variable based on another –findings supported by Sfard's (1987) results regarding the overwhelming predominance of operational/process conceptions of functions among algebra students. Additional PME research on the role of programming and other related environments in algebra learning includes the studies with Logo (e.g., Hoyles, Sutherland & Evans, 1985), with BASIC (e.g., Thomas & Tall, 1986), with software for exploring structured equivalent expressions (Thompson & Thompson, 1987), and with the CARAPACE environment featuring process-oriented approaches to algebra problem solving (Kieran, Boileau & Garançon, 1989). Despite the positive results of these early programming-oriented studies, many of the environments did not include the making of explicit connections with conventional algebraic concepts and notation.

Spreadsheets: A technological tool for bridging to algebraic forms and methods

Among the studies that have explored the use of spreadsheets as a means to introduce students to algebra are those conducted by Ainley (1996), where 11-year-olds used both the numerical-tabular and graphical facilities to model various problem situations. Building on their earlier research with spreadsheets (e.g., Sutherland & Rojano, 1993), Filloy, Rojano and Rubio (2001) have found that the spreadsheet serves as a bridging tool to algebra because it helps students both to create conceptual meaning for algebraic objects and operations and to move from focusing on a specific example to describing general relationships. For Dettori, Garuti and Lemut (2001), the functional orientation of spreadsheets allows for their effective use in the investigation of variation, but can cause difficulties later when solving equations or inequalities. When these researchers investigated 13- and 14-year-olds working on algebraic problems with spreadsheets, Dettori and colleagues were led to conclude that, "spreadsheets can start the journey of learning algebra, but do not have the tools to complete it" (p. 206). Balacheff (2001) has voiced similar concerns regarding the so-called algebra-like notation of spreadsheet environments. As well, Hershkowitz, Dreyfus, Ben-Zvi, Friedlander, Hadas, Resnick, Tabach and Schwarz (2002) have described how students tend, in the spreadsheet (tabular) environment, to generalize recursively rather than explicitly, which makes it more difficult for them to generate closed-form algebraic rules for patterns.

Graphing calculators and other related technological tools

Schwarz and Hershkowitz (1999) have investigated 9^{th} graders' learning of the concept of mathematical function in an environment that included problem situations, graphing calculators and multi-representational software tools, and where students were encouraged to make their own decisions about which representations to use, when and how to link the representations, and in which medium to work. A questionnaire was presented that included questions on the interpretation of graphical representations, the generation of graphical representations, and the relation of certain graphs to algebraic representations. The researchers found that the students: (a) Often used the prototypical linear and quadratic functions, but did not consider them as exclusive (for persistence of linear models in students' reasoning, see De Bock, Verschaffel & Janssens, 2002, as well as findings from earlier research on functional thinking, e.g., Markovits, Eylon & Bruckheimer, 1986), (b) Used prototypes as levers to handle a variety of other examples, (c) Articulated justifications often accounting for context, and (d) Understood functions' attributes. Schwarz and Hershkowitz suggest that the numerous manipulative experiences that were afforded by the graphing calculators and multi-representational software influenced positively students' understanding of the functions' attributes.

In contrast, Cavanagh and Mitchelmore (2000) have noted the presence of several misconceptions among their interviewees as they carried out various graphical tasks: A tendency to accept the graphical image uncritically, without attempting to relate it to other symbolic or numerical information; a poor understanding of the concept of scale; and an inadequate grasp of accuracy and approximation. Furthermore, in research reported by Hershkowitz and Kieran (2001) involving 16-year-olds working on a function-based problem situation integrating the graphing calculator as a tool, students were found to go from entering lists to a graphical representation by means of the regression tool, without ever seeing or having to examine the algebraic representation of the situation –a phenomenon that, according to the researchers, may suggest some caution in the way in which algebra instruction that integrates the use of graphing calculators is orchestrated. Despite what appear to be mixed results emerging from the above studies regarding students' understanding of graphical representations, a large majority of the research investigating the impact of graphing technology tools have reported significant benefits with respect to students' conceptualizing of functions, especially when these tools are used over long periods of time (e.g., Streun, Harskamp & Suhre, 2000).

Word problems and multiple representations

Much of the algebra research involving technological tools has focused on the solving of word problems by means of multiple representations. For example, Huntley, Rasmussen, Villarubi, Sangtong and Fey (2000) have found that, when students were able to use context clues and had graphing calculators available, those who were not strong in symbol-manipulation skills could outperform symbolically-capable students when the tasks required formulation and interpretation of situations. The impact of the availability of a range of representations on students' solving of complex contextual problems has also been demonstrated by Molyneux-Hodgson, Rojano, Sutherland and Ursini (1999). In this cross-national study, the multiple representations supported algebra learning within the cultures of the schools in different ways: Mexican students learned to appreciate and use graphical and numerical representations; and English students, to make more sense of letter-symbolic representations.

The long-term impact of a problem-based, functional approach to the teaching of algebra, one that was supported by intensive use of multi-representation technology, was studied by Yerushalmy (2000). By means of classroom observations and clinical interviews with students over their three years of following such a course, Yerushalmy found that the representation of problem situations evolved as follows: From numbers as the only means of modeling, to intensive work with graphs and tables, to the use of more symbolic representations. Students also moved from analyzing patterns of numbers by watching the behavior of increments to more explicit, closed-form, formulations of functional relationships involving pairs of numbers. Yerushalmy noted as well that students took a rather long time to appreciate algebraic symbols, and that they continued to prefer to use the situation as the source for their answers.

Further research (Yerushalmy & Shternberg, 2001), as well as the long-term observations of Chazan (2000), has led Yerushalmy (2000) to argue that, “a curricular sequence of algebra based on functions might have to address more subtleties and transitions than initially expected ... and that the complexity of helping students to value algebraic symbols may require more than bridging between representations” (p. 145). The results from a follow-up study suggested to Gilead and Yerushalmy (2001) that graphical tools may be useful as an exploratory support only for canonical problems (i.e., where y-intercept and slope are both given).

Computer algebra systems and other structured symbol manipulation environments

Recent research in the French *didactique* tradition (e.g., Guin & Trouche, 1999; Artigue 2002; Lagrange, 2003) has shown that technological tools, such as Computer Algebra Systems (CAS), can promote both conceptual and technical growth in mathematics, as long as the technical aspects are not ignored. More specifically, Lagrange (2000), adapting a framework developed by Chevallard (1999), has elaborated the notion that technique is the bridge between task and theory. In other words, as students develop techniques in response to certain tasks, they engage in a process of theory building. Instrumentation theory, which articulates the relation between tool use and conceptual development (Trouche, 2000), has begun to be applied to the learning of algebra (e.g., Drijvers, 2003).

The introduction of CAS use in high school algebra classes has been accompanied by research interest in students’ conceptions of equivalence. Ball, Pierce and Stacey (2003) have recently reported that recognizing equivalence, even in simple cases, is a significant obstacle for students. According to the researchers, “The ability to recognize equivalent algebraic expressions quickly and confidently is important for doing mathematics in an intelligent partnership with computer algebra; it is also a key aspect of algebraic expectation, the algebraic skill that parallels numeric expectation” (p. 4-16). They add that attention to equivalence is likely to take on new importance in future curricula (in this regard, see also Kieran & Saldanha, 2005). Despite its being a core idea in algebra, only a few of the earlier studies explicitly addressed students' notions of equivalence (e.g., Kieran, 1984; Linchevski & Vinner, 1990; Steinberg, Sleeman & Ktorza, 1990).

Another example of some of the most recent research in this area is the work of Cerulli and Mariotti (2001). These researchers have reported on a teaching experiment that involved 9^{th} grade classes (15-year-olds) and the algebra microworld, *l'Algebrista*, created by Cerulli (2004). Their aim was to develop in students a theoretical perspective on algebraic manipulation, by basing activity on proving and the concept of equivalence relation. The written justifications that students provided of their equivalence transformations, as seen in the iconography of the axiom buttons that were mediating their thinking, suggested to the researchers that a "proving" approach to equivalence was yielding positive results with respect to students' learning to solve equations and produce equivalent expressions.

Additional remarks

The research reviewed in this section has focused on the strands of work that first began to be seen in PME research from about the mid-1980s and that have continued to the present day. As was noted, one of the main influences on the research of this period has been the advent of technological tools that could be applied to the learning of algebra. Not only did this phenomenon encourage the movement of the study of functions with their multiple representations toward the domain of school algebra, it also succeeded in broadening the sources of meaning making that algebra students might invoke. Algebra became more than the study of equations and equation solving; it grew to encompass functions (and their representations) and the study of change, as well as include real world situations that could be modeled by these functions. In addition to the several research studies carried out on these themes, generalization also became a focus of research attention.

As was mentioned in the concluding remarks of the previous section, a socio-cultural theoretical perspective had begun to permeate PME research in the 1990s. The combination of digital tools being used in algebra learning, and the socio-cultural lens being applied to both the design of learning studies and the analysis of the resulting data, led to new questions about the nature of learning with these tools and the roles of the different participants in the classroom. Kaput (1992) pointed out: "[With these tools] the locus of social authority becomes more diffuse; provision must be made for students to generate, refine, and prove conjectures; the teacher must routinely negotiate between student-generated mathematics and the teacher's curricular agenda" (p. 548). Moreover, as emphasized by Hoyles (2002): "Tools can no longer be regarded as neutral players in the process of meaning making" (p. 275). There is no question but that the face of PME research in algebra was being indelibly altered by these forces. Not only was the content of algebra being widened by functional and generalization-based approaches, but also the theoretical and technological lenses that were being used to design and interpret research served to complexify analyses of the sources from which students might derive meaning in algebra. Additional contributing factors are discussed in the upcoming section.

THEME-GROUP THREE: ALGEBRAIC THINKING AMONG ELEMENTARY SCHOOL STUDENTS, A FOCUS ON THE ALGEBRA TEACHER/TEACHING, AND DYNAMIC MODELING OF PHYSICAL SITUATIONS AND OTHER DYNAMIC ALGEBRA ENVIRONMENTS

With the extension of equation-based algebra to include functions and the modeling of real world situations by various functional representations, the notion that algebra could be made accessible to more students (Chazan, 1996), even elementary school students (Kaput, 1995), was soon elaborated. Research on the development of algebraic thinking at the elementary school level began to be presented at PME during the mid-1990s. At about the same time, there was a significant increase in the amount of research involving the teacher of algebra. The learner has tended to be the main focus of attention of algebra researchers throughout the years of PME's existence; nevertheless, a large number of studies devoted to the algebra teacher and to the practice of algebra teaching have, of late, also been carried out.

As well, further developments in technology-related algebra research were seen in the mid-1990s in studies involving the integration of dynamic environments in algebra learning and the exploration and modeling of physical situations with technological tools. New theoretical perspectives soon evolved in an attempt to more adequately take into account the diverse dimensions of algebra learning that were being studied, in particular the roles being played by gestures, bodily movement, and language.

Algebraic thinking among elementary school students

The emerging body of research on an area that has come to be known as *Early Algebra* was reflected in the holding of a Research Forum on Early Algebra at the 2001 PME (Ainley, 2001). While the question of whether the study of algebra should be spread over all of grades 2 to 12 has been debated since the 1960s according to Davis (1985), the research focus on the algebraic thinking of the elementary age student is quite new.

As had been suggested by the algebra research findings emanating from the previous two decades of PME studies carried out with pupils beginning high school algebra, students operating in an arithmetic frame of reference tend not to see the relational aspects of operations; their focus is on calculating. Thus, considerable adjustment is required in learning, for example, to compare expressions for equivalence and to view the equal sign as other than a signal to compute an answer. The many research studies that centered on the difficulties involved in moving from an arithmetic to an algebraic form of reasoning (e.g., Kieran, 1979; Booth, 1981, 1984; Lee & Wheeler, 1989; Wagner & Kieran, 1989; Herscovics &

Linchevski, 1991, 1994; Kieran, 1992; Linchevski, 1995) have, by extension, provided both the motivation for thinking about beginning algebraic explorations in elementary school and the foundation for some of the recent research on the emergence of algebraic thinking among elementary school students.

Some of the principal Early Algebra themes that have been developed in the research by members of the PME community include: Relational thinking about numeric equalities, symbolizing relationships among quantities, working with equations, developing functional thinking, and fostering an understanding of mathematical properties. Among the studies focusing on the emergence of relational views of numeric equalities is the research of Fujii (2003), who has introduced young children to algebraic thinking through generalizable numerical expressions, using numbers as *quasi-variables* (e.g., number sentences such as 78 – 49 + 49 = 78, which are true whatever number is taken away and then added back). A related study by Ishida and Sanjii (2002) has found that the understanding of structure increases from grades 4 to 6 (from age 10 to age 12), which suggested to the researchers that as students grow older they come to understand the role of a mathematical expression from the point of view of both structure and generality. The multi-year research program of Zack (1995, 1997, 2002) has provided evidence of the ways in which 10- and 11-year-olds learn to reason about, prove, and represent symbolically the complex generalizations underlying the structure of problems such as "squares on a chessboard" or "ferret tunnels / handshakes / polygon diagonals."

Another strand in Early Algebra research has been the symbolizing of relationships among quantities. According to the approach developed by Davydov and his colleagues (see Schmittau & Morris, 2004), it is not experience with number that serves as the basis for learning algebra; rather its foundation rests on relationships between quantities that children search out across contextualized situations. Dougherty and Slovin (2004), who integrated Davydov's approach within their *Measure Up* project, have reported that 3rd graders can learn to use algebraic symbols and diagrams meaningfully by working with measurement situations.

The research findings of Schliemann, Carraher, Brizuela, Earnest, Goodrow, Lara-Roth and Peled (2003) have led these researchers to argue that students as young as 9 and 10 years of age are able to develop an enlarged sense of the equality sign, represent unknown quantities with a letter, represent relations with variables, work with unknowns, write equations, and even solve letter-symbolic linear equations. Warren (2003) has, however, noted that 8- and 9-year-olds can experience difficulty in handling problems with unknowns. Nevertheless, Da Rocha Falcão, Brito Lima, De Araújo, Lins Lessa and Oliveira Osório (2000) have also reported that activity involving symbolic representation, manipulation of relations of difference and equality, and manipulation of unknowns is feasible with 10- to 12-year-olds, provided that the passage from known to unknown quantities is handled carefully and with adequate preparatory activities that include verbalization.

Although pattern finding in single-variable situations is now fairly common in elementary curricula according to Blanton and Kaput (2004), they argue that elementary school programs should also include functional thinking within algebraic reasoning activities. They base this argument on their findings that students as young as those in kindergarten can engage in covariational thinking, that 1st graders can describe how quantities correspond, and that 3rd graders can use letters as variables (see also Warren, 2005).

While certain proponents of algebraic thinking in elementary school support the early introduction of the letter-symbolic (e.g., Schliemann et al., 2003); others hold the opposite position (e.g., Fujii, 2003). Thus, definitions of algebraic thinking in the early grades rarely take both perspectives into consideration. Kieran (1996) has characterized the activities of school algebra according to three types: Generational, transformational, and global/meta-level. While this characterization was initially designed with the secondary school student in mind, the global/meta-level activities of algebra, which provide context and a sense of purpose for letter-symbolic work in algebra, afford a basis for delineating algebraic thinking that is broad enough to include the various positions regarding the introduction of symbolic work at the elementary school level:

> *Algebraic thinking in the early grades* involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool, but which are not exclusive to algebra, and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. (Kieran, 2004, p. 149)

A potential advantage of appropriating this definition of algebraic thinking for Early Algebra is the bridge building it affords between research in algebra learning involving the elementary school student and that of the older student at the secondary level. In addition, this characterization captures the main thrust of the algebra research to date at the elementary school level, research that has focused on the analysis of relationships between quantities (e.g., Dougherty, 2003), the noticing of structure within activity on numerical properties (e.g., Carpenter, Franke & Levi, 2003), the study of change (e.g., Noble, Nemirovsky, Wright & Tierney, 2001), generalizing (e.g., Fujii, 2003), and the solving of problems with a focus on relations (e.g., Zack, 2002).

Algebra teaching and the algebra teacher

While it could be argued that all research on algebra learning implicitly involves teaching, it was not until the 1990s that algebra teaching and the algebra teacher became a focus of analysis by PME researchers. In 1987, Shaughnessy (1987) remarked in his reaction to the then-current PME research on teachers and teaching that most of this research remained quite general and had yet to feature the teaching of specific mathematical content. Coincidentally, the following year, Even

(1988) presented a research report at PME on pre-service teachers' conceptions of the relationships between functions and equations. That these prospective teachers held a limited view of functions as equations only, was not so surprising, given the rather recent integration of functions into the domain of school algebra. What was remarkable about this research report was that it signaled the gradual emergence, during the ensuing years, of PME research on algebra teaching and the algebra teacher (e.g., Even, 1990; Robinson, Even & Tirosh, 1992; Pence, 1994) and the beginnings of the growth of theoretical frames of reference suitable for analyzing data on the algebra teacher and teaching (see the following research reports for some of these theoretical frameworks: Solomon & Nemirovsky, 1999; Coles, 2001; Boaler, 2003; Doerr, 2003; English & Doerr, 2003; Simmt, Davis, Gordon & Towers, 2003).

The research that foregrounds the teacher of algebra and/or the teaching of algebra can be grouped into the following three categories: Studies dealing with the practicing teacher in the algebra classroom, those conducted within professional development or in-service training programs, and those involving the preservice teacher of algebra. The research on algebra teaching and the practicing algebra teacher has produced studies related to teachers' knowledge of students' ways of thinking (e.g., Hadjidemetriou & Williams, 2002), as well as the various influences on teachers' practice (e.g., Doerr & Zangor, 2000; Guin & Trouche, 2000).

Teachers' knowledge of students' ways of thinking in the context of algebra has been the focus of research by Tirosh, Even and Robinson (1998). Their research has led them to argue for more of the following kind of study: "While one can use the mathematics education literature in order to raise teachers' sensitivity to students' ways of thinking, ... this literature does not offer enough information and discussion regarding the impact of various approaches and teaching methods related to this conception; ... it is important for teachers to be acquainted with various teaching methods and to be aware of their pros and cons in different contexts with different teaching aims, and with different students" (pp. 62-63).

From her analyses of traditional and reform-oriented classroom teaching, Boaler (2003) has found that, contrary to the common perception that reform-oriented classrooms are less teacher-centered, teachers actually spend more time questioning the whole class in reform classes. However, it is not just the amount of time that is different, it is the nature of the teacher talk. According to Boaler, "the teachers in the traditional classes *gave* students a lot of information, while the teachers of the reform classes chose to *draw* information out of students, by presenting problems and asking students questions" (p. 1-4).

The research that has been conducted within the context of professional development or in-service training of algebra teachers has focused on two areas in particular, the algebra content knowledge and beliefs of teachers (e.g., Barkai, Tsamir, Tirosh & Dreyfus, 2002), as well as the integration of new curricula, teaching approaches, and technological environments into the algebra classroom (e.g., Stacey, Kendal & Pierce, 2002; Cedillo & Kieran, 2003; Zehavi, 2004; Thomas & Hong, 2005). An example is drawn from the ArAl project (Malara, 2003), an in-service teacher education project devoted to the renewal of the teaching of arithmetic and algebra in the 6th to 8th grades. The research explicitly included the classroom teacher as a researcher and emphasized knowledge of pedagogical content, the teacher's role in the teaching of mathematics, the impact of his/her personality, as well as social issues within the class group. A second example is provided by the research of Ruthven and Hennessy (2002), who interviewed groups of teachers from the mathematics departments of seven secondary schools in England –teachers involved in a project on integrating new technologies into their teaching. The researchers found that the teachers tended to embrace a broad use of computing tools in the teaching of mathematics at these grade levels, including spreadsheets, graphical software, and Logo programming, in addition to courseware that had been designed to teach or test a particular mathematical topic.

Research involving the preservice teacher of algebra has inquired into, for instance, the role of beliefs and attitudes (e.g., Vermeulen, 2000), as well as the nature of the content knowledge of the preservice trainee (e.g., Zazkis & Liljedahl, 2002; Presmeg & Nenduradu, 2005). For example, Van Dooren, Verschaffel and Onghena (2002, 2003), who have studied preservice teachers' preferential strategies for solving arithmetic and algebra word problems, found evidence that future secondary school mathematics teachers tend to use algebraic methods for solving even very easy problems that could have been handled more appropriately with arithmetic methods. Half of the primary school preservice teachers in the study could switch flexibly between arithmetic and algebraic methods, while the other half had difficulty with algebraic methods. It was also found that the methods the teachers used as individuals were strongly correlated with that which they would expect their future students to use and by which they would evaluate student work. Similar results have been reported by Schmidt and Bednarz (1997).

Dynamic modeling of physical situations and other dynamic algebra environments

Another rather recent strand in PME algebra research involves modeling activity that includes actual physical artifacts and/or technological modeling tools. The phenomena that students have been observed modeling within various research studies include cars moving along tracks (Schnepp & Nemirovsky, 2001) and devices with gears of different sizes (Bartolini Bussi, 1995; Meira, 1998). For instance, Schnepp and Nemirovsky (2001) have reported on an environment where a computer is linked to miniature cars on parallel linear tracks (or to a miniature stationary bike) and where the user can construct a graph on a computer, which in turn communicates with a motor that moves the mechanical device according to the graphical specifications. In this example, it is noted that the activity involves the inverse of modeling, that is, the direct control of a mathematical model in order to manipulate a physical object (see related examples in Kaput & Roschelle, 1997).

Arzarello and Robutti (2001) have described a study introducing 9th graders to algebra and involving the interpretation of body-motion graphs with a symbolic-graphing calculator connected to a motion sensor (a calculator-based ranger – CBR). Initially students were encouraged to try various running patterns in order to create different graphs. The continuous nature of the CBR graphing allowed students to test conjectures in a direct manner, controllable by their own physical movement. Arzarello and Robutti observed that, "students' cognitive activity passes through a complex evolution, which starts in their bodily experience (namely, running in the corridor), goes on with the evocation of the just lived experience through gestures and words, continues connecting it with the data representation, and culminates with the use of algebraic language to write down the relationships between the quantities involved in the experiment" (p. 39).

Other such environments offering *dynamic control*, that is, "the direct manipulation of an object or a representation of a mathematical object –the manipulation being of a continuous nature, or discrete" (Kieran & Yerushalmy, 2004, p. 120), include those where direct manipulation is achieved by means of slidergraphs (Zbiek & Heid, 2001), sliders (Schwartz & Yerushalmy, 1996), dragging facilities (Borba, 1993), and so on. The research being conducted in these environments has led to the development of new theoretical perspectives where bodily experience and other factors that could be said to be exterior to the mathematics are considered to play vital roles in the creation of meaning for algebraic objects.

Remarks on the recent emergence of new theoretical perspectives

An indication of the growth of research interest in this new source of meaning in algebra learning was the holding of the first Research Forum on the role of perceptuo-motor activity in mathematical learning, organized by Nemirovsky (2003), at the 2003 PME conference. The research findings presented at that forum suggested a complex relationship between gestures and words that allows students to make sense of algebraic expressions (Radford, Demers, Guzmán & Cerulli, 2003) and illustrated how motion experiences contribute to students' meaning making in algebra (Robutti & Arzarello, 2003). As well, Ferrara (2003), using a framework based on the theory of Lakoff and Núñez (2000), described how metaphors can serve as vehicles of algebraic knowledge and means of sharing knowledge (see also Sfard, 1994; Boero, Bazzini & Garuti, 2001). While these viewpoints with their emphases on discourse and activity clearly intersect with discursive frames of reference (e.g., Kieran, Forman & Sfard, 2001) and the construct of "situated abstraction" (Noss & Hoyles, 1996), they also add a perspective previously unseen in PME algebra learning research with their attention to gestures, bodily experience, and metaphors as a source of meaning making in algebra.

TRAVERSING THE VARIOUS THEME-GROUPS THAT HAVE EMERGED IN PME ALGEBRA RESEARCH: A BROADENING OF SOURCES OF MEANING

The question of meaning lies at the heart of PME research in algebra. The various shifts in emphasis over the years from the letter-symbolic through multiple representations to algebra learning in dynamic environments hint at the evolution that has occurred with respect to sources of meaning in algebra learning research. Researchers of the PME community have long been and continue to be interested in issues of meaning making in mathematics, in general, and algebra, in particular (e.g., Kaput, 1989; Sackur, Drouhard, Maurel & Pécal, 1997; Arzarello, Bazzini & Chiappini, 2001; Filloy, Rojano & Rubio, 2001; Lins, 2001). For example, Kaput (1989) has described four sources of meaning in mathematics, which he divides into two complementary categories: *Referential extension*, which consists of translations between mathematical representation systems and translations between mathematical and nonmathematical systems; and *consolidation*, which consists of pattern and syntax learning through transforming and operating within a representation system and building conceptual entities through reifying actions and procedures. Clearly related to this latter category, consolidation, are the various theoretical process-object distinctions that have been developed with respect to the construction of meaning in mathematics (e.g., Dubinsky, 1991; Sfard, 1991; Gray & Tall, 1994), as well as the recent models of abstraction (e.g., Hershkowitz, Schwarz & Dreyfus, 2001; Schwarz, Hershkowitz & Dreyfus, 2002).

Another framework for thinking about meaning, which has been used in several algebra studies, is that of Frege's semiotic triangle (e.g., Sackur et al., 1997; Arzarello et al., 2001; Bazzini, Boero & Garuti, 2001; Drouhard & Teppo, 2004), in particular his distinction between sense and denotation[2]. For example, in their theoretical analyses of the meaning of symbolic expressions in algebra, Arzarello et al. (2001) explain that, "All possible senses of an expression constitute its so called intensional aspects, while its denotation within a universe represents its so called extensional aspect. ... The official semantics used in mathematics, and particularly in algebra, cuts off all intensional aspects, insofar as it is based on the assumption of the extensionality axiom (two sets are equal if they contain the same elements, independently from the way they are described or produced)" (p. 64). However, as they and others (e.g., Sfard & Linchevki, 1994) have pointed out, intensional aspects are crucial to algebra learning because of the difficulties students experience in mastering the invariance of denotation with respect to sense.

Yet another perspective is that of Kirshner (2001), who has characterized the approaches to meaning making in elementary algebra in terms of the structural and the referential. According to Kirshner, "the structural approach builds meaning internally from the connections generated within a syntactically constructed system, while the referential approach imports meaning into the symbol system from external domains of reference" (p. 84).

More recently, the various ways of thinking about meaning making in algebra have been reconceptualized to suggest a threefold approach. For Radford (2004), meaning in school algebra is produced in the "crossroads of diverse semiotic mathematical and non-mathematical systems" (p. 1-163) and is, according to Radford, deemed to come from three primary sources: (i) The algebraic "structure" itself, (ii) The problem context, and (iii) The exterior of the problem context (e.g.,

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[2] Within the Fregean theoretical framework involving sense and denotation, the expressions $4x + 2$ and $2(2x + 1)$ have different senses, but denote the same functional object.

linguistic activity, gestures and bodily language, metaphors, lived experience, image building, etc.).

A comparison of Radford's threefold classification with the more general, triple perspective on "meaning in mathematics education" offered by Noss and Hoyles (1996) –*meanings from mathematical objects*, *meanings from problem solving*, and *meanings constructed by the individual learner*– as well as a consideration of the existing body of algebra research findings, suggests that Radford's first source of meaning, the "algebraic structure itself," could usefully be expanded further to take into account the meaning building that is derived from other mathematical objects, such as graphical forms. This leads to my following adaptation of Radford's schema for algebraic meaning and its sources (see Figure 1). A limited elaboration of these four sources of meaning will ensue in the upcoming paragraphs, with brief connections drawn to the themes that have emerged in PME algebra research over the past 30 years.

Figure 1: Sources of meaning in algebra (adapted from Radford, 2004).

1. Meaning from within mathematics:
   - 1(a). Meaning from the algebraic structure itself, involving the letter-symbolic form.
   - 1(b). Meaning from multiple representations.
2. Meaning from the problem context.
3. Meaning derived from that which is exterior to the mathematics/problem context (e.g., linguistic activity, gestures and bodily language, metaphors, lived experience, image building, etc.).

Meaning from the algebraic structure itself, involving the letter-symbolic form

Booth (1989) has argued that, "our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations which distinguish allowable transformations from those that are not. These structural properties constitute the *semantic* aspects of algebra. ... The essential feature of algebraic representation and symbol manipulation, then, is that it should *proceed from* an understanding of the semantics or referential meanings that underlie it." (pp. 57-58). In other words, "the sense of meaningfulness comes with the ability of 'seeing' abstract ideas hidden behind the symbols" (Sfard & Linchevski, 1994, p. 224), that is, the symbols become transparent. This aspect of algebra is sometimes referred to as its internal semantics.

This structural source of meaning not only links letter-symbolic representations to their numerical foundations but also interweaves the forms of algebra, its equivalences, and its property-based transformational activity (as noted, for example, in the research of Cerulli and Mariotti, 2001, where "the use of axioms becomes the only way to state the equivalence between two expressions, whilst numerical verification becomes the main way to prove that two expressions are not equivalent", p. 184). While "grasping the structure of expressions" –a phrase often used in research dealing with the structural source of meaning in algebra– is rarely spelled out in detail, researchers have nevertheless offered definitions for related phrases, for example, *structure* (Hoch & Dreyfus, 2004), *structure sense* (Linchevski & Livneh, 1999), *symbol awareness* (MacGregor & Price, 1999; see also the framework for *algebraic expectation* by Pierce & Stacey, 2001), and *symbol sense* (Arcavi, 1994). Even though the nature of the meaning that students draw from algebraic structure can be elusive (e.g., Demby, 1997), this source of meaning is considered by many mathematics educators and researchers to be fundamental to algebra learning.

Meaning from multiple representations

Kaput (1989) has argued that the problem of student learning in algebra is compounded by (a) the inherent difficulties in dealing with the highly concise and implicit syntax of formal algebraic symbols and (b) the lack of linkages to other representations that might provide feedback on the appropriate actions taken. As a consequence, he has promoted the kind of mathematical meaning building that has its source in *translations between mathematical representations systems*. While tables, graphs, and equations can all be used to display binary relations, a critical difference, according to Kaput, between tables and graphs is that, "a graph engages our gestalt-producing ability, which allows us to consolidate a binary (and especially a functional) quantitative relationship into a single graphical entity –a curve or line" (Kaput, 1989, p. 172, parentheses in the original).

Multi-representational sources of meaning and functional approaches to the teaching of algebra clearly go hand-in-hand. It is noted that various representations are generally seen as mediators between certain characteristics of the representing notation and features of the world being modeled –a source of meaning to be treated in the next subsection. However, in the context of within-mathematical sources of meaning, it is the interrelationships among the various mathematical representations themselves that are considered to support meaning building in algebra. The opportunity to coordinate objects and actions within two different representations, such as the graphical and the letter-symbolic, has been found to be helpful in creating meaning in algebra (e.g., Fey, 1989; Romberg, Fennema & Carpenter, 1993; Yerushalmy & Schwartz, 1993), Nevertheless, inferring from visualization of graphs has also been shown to be potentially misleading (e.g., Goldenberg, 1988). Difficulties arise as well from the need to be flexible and competent in translating back and forth between visual and analytic reasoning based on the various representations (e.g., Arcavi, 2003; Amit & Fried, 2005).

Researchers whose studies have integrated computers and calculators involving symbolic manipulation, with graphical and tabular representations (e.g., Guin & Trouche, 1999; Artigue, 2002; Lagrange, 2003), have argued that the emergence of meaning for both symbolic forms and graphical representations, and the relation between the two, cannot be achieved without the simultaneous development of techniques that are both mathematical and technological. However, as has been pointed out by Yerushalmy and Chazan (2002), multiple representation technology brings with it not only access to representations that are different from the letter-symbolic, but also added complexity due to the potential functional interpretations of the algebraic objects being represented. The question of the meaning given to processes such as equation solving, when the meaning for the equation being manipulated is related to a functional graphical representation, is one where further research is indeed needed.

Meaning from the problem context

In contrast to the internal semantics of algebra as a site for meaning making, as argued by Booth (1989), the external semantics of a problem permit the algebra learner to fuse symbols and notations with events and situations, thereby creating an external meaning for certain objects and processes of algebra. A large body of research in algebra learning holds that problem-solving contexts are foundational to the emergence and evolution of algebraic reasoning (e.g., Bednarz & Janvier, 1996; Bell, 1996). This stance is based to a certain extent on historical grounds whereby problem solving made a major contribution to the development of algebra, as well as on the gradual growth in the status of algebra as a privileged tool for expressing general methods for solving whole classes of problems. Additional arguments are based on relevance and purpose. Despite these positions in favor of the role of problem solving as a source of meaning building in algebra, it has also been emphasized (e.g., Pirie & Martin, 1997; Balacheff, 2001) that there are significant epistemological issues associated with the whole idea of using word problems for creating algebraic meaning making. However, a broader perspective on problem solving has been advanced by Bell (1996):

> For introducing and developing algebra, I understand problem solving to refer to the solving of problems by the forming and solving of equations; this is the narrow sense of the term. But the essential mathematical activity is that of exploring problems in an open way, extending and developing them in the search for more results and more general ones. Hence [all algebraic learning] ... is based on problem explorations. This is the broad sense of the term. (p. 167)

Included within this source of algebra meaning making are “real problems,” that is, those that involve the modeling of situations with some mathematical representation, applying mathematical techniques to solve them, and translating the results back to the original situation. Much of the current modeling activity that occurs in algebra classes uses actual physical artifacts and/or technological modeling tools as an integral part of the activity. In these cases, the contextual aspect of the situation is very much intertwined with gestures and body language in the process of constructing a mathematical model; this combines two sources of meaning, one derived from the problem context and the other which is exterior to the context and which is discussed directly below.

Meaning derived from that which is exterior to the mathematics/problem context

As pointed out in earlier theoretical remarks related to the research directions that began to emerge in PME during the mid-1990s, a recent approach by algebra researchers to thinking about meaning making concerns the sources of meaning that are exterior to both the mathematics and the problem situation. (It is noted that much of this meaning building that is "exterior" to both the mathematics and the problem situation is related to that which is uniquely human, and thus is clearly "interior" in a very real sense.) Research with an eye to students' processes of meaning production in terms of the way diverse resources such as gestures, bodily movement, words, and artifacts become interwoven during mathematical activity has attracted much interest (e.g., Arzarello & Robutti, 2001; Boero, Bazzini & Garuti, 2001; Nemirovsky, 2003; Radford et al., 2003; Robutti & Arzarello, 2003; Radford, Bardini, Sabena, Diallo & Simbagoye, 2005). Past studies of students' ways of thinking in algebra had always suggested that they brought more to bear on their learning of algebra than were accounted for by the theories available at the time; thus, the explicit focus in recent algebra research on bodily activity, language, and past lived experience as a source of meaning is a natural evolution and one that merits further theoretical and empirical attention.

Based on his seminal studies of students' production of (oral and written) signs and the meanings they ascribe to them as they engage in the construction of expressions of mathematical generality, Radford (2000) has observed that:

> Students, at the very beginning, tend to have recourse to other experiential aspects more accessible to them than the structural one. ... Novice students bring meanings from other domains (not necessarily mathematical domains) into the realm of algebra. Hence it seems to us, one of the didactic questions with which to deal is ... that of the understanding of how those non-algebraic meanings are progressively transformed by the students up to the point to attain the standards of the complex algebraic meanings of contemporary school mathematics. (p. 240)

With the broadening of sources of meaning in algebra to include that which is derived from the exterior of the mathematics/problem context, Radford's question is an obvious one for future research and one with which to bring this chapter to a close, that is, how do students come to integrate and transform the meanings they derive from the various sources, both mathematical and non-mathematical, so as to "attain the standards of the complex algebraic meanings of contemporary school mathematics." This fundamental question, within the intricacy of a multiple-sources-of-meaning perspective, offers a rich springboard for future PME research in algebra.

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AFFILIATION

Carolyn Kieran
Département de Mathématiques
Université du Québec à Montréal
C.P. 8888, succ. Centre-Ville
Montréal, QC, H3C 3P8 (Canada)
kieran.carolyn@uqam.ca