This book may be used for research, graduate and undergraduate teacher education, and teacher development. It presents an integrated set of studies of a heterogeneously grouped class of twenty-one nine-year olds, engaged in exploring fraction ideas prior to classroom instruction under conditions that supported investigation, collaboration and argumentation. It demonstrates with text and video narrative how young children can reason about mathematics in surprisingly sophisticated ways when provided the opportunity to do so in the proper classroom environment. In this volume, fourth grade students’ reasoning about fraction concepts is described through careful analysis and accompanying video excerpts showcasing the variety and originality of their thinking. These children will serve as an inspiration for educators to encourage the development of reasoning and argumentation in their students as part of a mathematics curriculum designed to produce critical thinkers.
Children’s Reasoning While Building Fraction Ideas
MATHEMATICS TEACHING AND LEARNING

Volume 3

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For further information:
Children’s Reasoning While Building Fraction Ideas

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When I began my work as Principal in a suburban New Jersey elementary school, I brought with me a passion to work with teachers to improve the learning and teaching of mathematics that went on in our classrooms. In order to accomplish this goal, I knew that I would have to provide the teachers in my school with both support and long-term teacher development. After carefully investigating and considering potential partners for my work with teachers, I decided to contact my former mentors and colleagues at Rutgers University, my alma mater. Working with them inspired me, and I hoped that there would be a way to create an engaging professional environment that both encouraged my teaching staff to continue to learn and also supported research in the school’s classrooms.

During my first year as the principal, I contacted Professors Maher and Davis, my former mentors at Rutgers University, to discuss forming a university-school partnership that would help me to realize my dream of a dynamic staff, fully committed to teaching our students and also to their own learning. I believed by linking with the university faculty and their doctoral students, my teachers would have opportunities to work collaboratively with researchers and graduate students and become aware of new approaches and knowledge development. Setting up the staff development opportunities for teachers and providing regular support to teachers in their planning and implementing of classroom lessons had the potential to provide new opportunities for teachers at my school, offering them multiple ways to revisit their views and beliefs about mathematics learning. I felt that they would be inspired by the Rutgers colleagues and the approach that they would bring to our school. I anticipated the partnership with Rutgers would also provide an opportunity to do research in a real school setting, working with teachers and their students, and that the knowledge we discovered together would have the potential to improve mathematics teaching beyond the walls of my school in suburban New Jersey.

Our school/university partnership flourished over a ten-year period, and it took many forms. We began with graduate students who offered model lessons in two of our classrooms. To make this possible, arrangements needed to be made to free other third and fourth grade teachers to observe these model lessons that focused on student investigations and problem solving. I met with the teachers and, with occasional visits from Rutgers faculty, we all met to discuss the lessons and together plan new ones. I vividly remember the excitement and the stimulated discussions we held. There was always room in our discussions for multiple views, resulting in
insights for all to gain. Of course, there were some logistical issues to be solved, such as for additional teacher support as the teaching staff experimented with more open-ended activities and small group work. In addition, we needed to adjust scheduling, so that I could meet with all grade level groups on a rotating basis. I joined the teachers in discussions, planning, and occasionally as my time allowed, also as a partner in implementing the new lessons.

With the assistance of Rutgers University faculty and graduate students, we were successful in engaging the teachers as learners. They reveled in justifying their solutions and sharing their new insights with each other! These learning communities were also critical in providing further support for teachers as they returned to their classrooms and implemented these same problems for their students to solve. Teachers, returning to their grade level meetings, shared and discussed student work, noting the variety of ways their children went about solving the problems.

As teachers began to engage their students in more thoughtful mathematics activities, they found that forty minutes were not enough time for mathematics. Consequently, we modified the time allotted for mathematics instruction, enabling teachers to be able to have at least an hour each day, as well as the flexibility of extending that period even more when it was needed.

During the 1992–1993 school year, a fourth-grade teacher partnered with Carolyn Maher to engage her students in problem solving. Throughout that year, Researcher Maher and doctoral student Amy Martino implemented about 50 lessons, engaging the students in thoughtful mathematics. Graduate students, faculty, and visitors came in to observe the classes. The lessons were captured with two or three cameras. Arranging for extended class periods, I visited classes and engaged with the students who were eager to explain their ideas and their reasoning for their solutions. Having a more extensive mathematics period, coupled with the flexibility to extend it when necessary, enabled some extraordinary mathematical thinking among students to take place in that fourth-grade classroom.

I knew it was important to share the changes that we were making in mathematics in my school with key people in the district. In addition to talking about this at administrative meetings, I made sure to encourage the Superintendent and Curriculum Director to come in and observe our mathematics classrooms. I also held meetings with parents, where they too got to engage as learners, solving problems in much the same way as their children did. I attended the Board of Education meetings and shared video clips, so they could have a better understanding of how mathematics was becoming more meaningful in my school.

During the ten years I served as Principal, our teachers evolved to become advocates of collaborative student learning on open-ended problem tasks. They became more comfortable, and even enthusiastic on occasion, in sharing what they were doing with each other as well as with other teachers in the county and state. Together we attended local and state mathematics conferences and shared the changes in our mathematics teaching and learning.
Our teachers were inspired to continue learning by taking courses at Rutgers University and working towards advanced degrees in mathematics education. To help them in this endeavor, Rutgers University provided a graduate level course on campus that the teachers were able to attend immediately after school hours.

This school-university partnership continued for the entire ten years that I was Principal. This experience produced a profound impact on the way mathematics was taught in our school. When I left my school to become a K-12 Director of Mathematics and Science in another district, I felt that much had been accomplished and that thoughtful mathematics learning and teaching was being provided in all classrooms.
ACKNOWLEDGEMENTS

We are indebted to the fourth graders from whom we have learned so much and continue to learn from their work. We express our appreciation to the Colts Neck school district, parents, principal Dr. Judith Landis, and teacher Ms. Joan Phillips for opening their school and classroom to us. We are grateful for the careful analysis of video data from dissertation work of Rutgers Graduate School of Education doctoral students.
INTRODUCTION

This book may be used for research, graduate and undergraduate teacher education, and teacher development. It presents an integrated set of studies of a heterogeneously grouped class of twenty-one, nine-year olds, engaged in exploring fraction ideas prior to classroom instruction under conditions that supported investigation, collaboration and argumentation.

The intervention was based on the view that students should have many opportunities to display their conceptual understanding and developing knowledge of fraction ideas through their explorations, explanations and justifications, expressed with oral and written language, symbols, models and other constructions. Throughout the intervention, students were encouraged to share and support their ideas, question arguments that were unclear to them, revisit earlier ideas, and work together to resolve differences in solutions to strands of well-defined, mathematical tasks. The chapters report student activities over a three-month period and illustrate, in student problem solving, a range of mathematical practices: sense making, model building, collaboration, communication, reasoning, argumentation and justification.

The video narratives accompanying each chapter enable the viewer to follow students’ journeys in their attempts to construct understanding of fraction ideas with minimal researcher intervention. We call attention to the proof-like reasoning displayed by the students as they provide justifications for their solutions to strands of tasks (see Appendix A, this volume) and as they develop a deep understanding of basic fraction ideas. Also, we illustrate how researchers draw attention to knowledge that was unearthed in students’ arguments. During classroom sessions - as we invited students to revisit key ideas, concepts, and ways of reasoning as a classroom community - we traced students’ forms of reasoning as they worked to resolve certain obstacles and inconsistencies that arose in their problem solving. By challenging the students to themselves resolve the issues, we were able to identify the forms of reasoning that were used in producing justifications. We call attention to these proof-like forms of arguments in the chapters that follow.

As we observe student participation in building solutions to strands of tasks, it is possible to trace how the community determines the reasonableness of the arguments presented. One can also attend to researcher moves in facilitating ongoing student participation and collaboration. We call your attention to the evolution of student constructions of proof-like arguments and counterarguments.
as student reasoning naturally evolves, (e.g., Maher & Davis, 1995; Maher & Martino, 1996a; Van Ness, Chapter 16, this volume) sometimes unearthing faulty reasoning (e.g., Chapter 8, this volume) and misconceptions (e.g., Chapter 17, this volume). Our approach to establishing a context for learning is based on a constructivist perspective (see Davis, Maher, & Noddings, 1990). Researcher interventions were guided by a view of teaching from this perspective (Maher, 1998a). Student misconceptions or faulty reasoning when surfaced, became opportunities for researchers to invite the students, themselves, to resolve them. Hence, as misconceptions surfaced, we deliberately held back in correcting students. Instead, we intentionally chose to intervene in a more subtle way, that is, to implement a new, engaging task to challenge the students to address and resolve the cognitive obstacle (e.g., Chapters 17 & 18, this volume; Maher & Davis, 1995; Mueller, Yankelewitz, & Maher, 2010; Yankelewitz, Mueller, & Maher, 2010; Francisco & Maher, 2005).

Student misconceptions or cognitive obstacles were viewed in this intervention as faulty constructions that needed to be brought to the attention of the classroom community. From this perspective, it would be inappropriate for a researcher to correct a student when a claim that was made by a misconception surfaced. Consequently, researchers did not “correct” student errors or faulty reasoning. Rather, they designed new tasks to call attention to contradictions that surfaced, making it possible for all students to engage in its resolution through argumentation, supported by evidence. Hence, all students were challenged to analyze the reasoning that produced the faulty claim or reasoning, making possible opportunities for all students to build deeper knowledge.

We distinguish mistakes from misconceptions and faulty reasoning. We all make errors in calculations or in our failure to attend to the parameters of the task at hand. The intervention provided opportunities for the students to reveal and self-correct mistakes or through student-to-student or researcher-to-student questioning (e.g., Maher & Martino, 1996b; Davis, Maher, & Martino, 1992).

Notice, at the end of each chapter, we offer questions for discussion, calling attention to student reasoning and other key ideas. It is our hope that in discussions that follow, argumentation and debate, similar to that of the fourth-graders presented in this study, evolve. We recommend that prior to the discussion, participants work on the tasks, sharing their own problem solving with others. We offer the tasks and tools for continued learning.

In this volume, we deliberately avoid recommendations for practice. It is our hope that those responsible for fraction instruction take notice of the understandings that these young children bring to the classroom and build new knowledge of the potential that students have to build deep understandings and ways of reasoning under the conditions offered in the study. We recognize that the conditions for student learning established for these studies do have implications for practice in other grade levels, contexts and communities. Because we value the importance of student engagement in learning, we emphasize the importance of soliciting
and listening to the ideas of others, both in terms of problem definition, building convincing arguments for problem solutions, incorporating the input of classmates into problem solutions as appropriate, and writing up a problem solution in a way that is convincing to oneself and others, as well as preparing a formal presentation of a problem solution to peers (Maher, 2005). These are life-long skills, applicable to other areas of learning and valued in a creative work environment (Maher, Francisco, & Palius, 2012).

While we do not offer an instructional program for teachers, we report implications for practice in other related publications (e.g., Maher, 1998a; Maher et al, 2014; Maher, Landis, & Palius, 2010; Mueller, Yankelewitz, & Maher, 2014). We are hopeful that the research reported in this book will serve as a guide for teachers and teacher educators in building effective instructional programs, attentive to children’s reasoning while building fraction ideas.

The data reported in the chapters that follow were collected in a heterogeneously grouped fourth grade-class in the district of Colts Neck, NJ (USA) in the academic year 1993–1994. Researcher Carolyn A. Maher, assisted by doctoral student, Amy Martino, led the fourth-grade classroom sessions, working with the students of classroom teacher, Joan Phillips, who shared her classroom and students with the research team. Principal Dr. Judith H. Landis provided the conditions for the study (e.g., parental permission, extended class time). She invited parents to join meetings to learn about the intervention project, and a classroom session to experience, with their children the problem-solving activities. The individual chapters are grounded in an extensive body of research that has been conducted as part of cross-sectional and longitudinal studies of the development of students’ mathematical ideas and ways of reasoning. The original data, including the video tapes, students’ work, transcripts, and detailed narrative analyses of students’ learning are archived in the Rutgers University Video Mosaic Collaborative (www.videomosaic.org), which is available worldwide, open-source. Several dissertations were produced from the study and these are listed in the references and again in Appendix B. Detailed information about the overall design and an overview of the chapters follow in Chapter 1.

The studies are based on extensive analyses of rich video and audio data, researcher notes, and transcripts from an intervention project that was funded by the following National Science Foundation grants: *Synthesizing Video Data on Students’ Mathematical Reasoning* (DRL-0723475, directed by Carolyn A. Maher); *Cyber-Enabled Design Research to Enhance Teachers’ Critical Thinking Using a Major Video Collection* (DRL-0822204, directed by Carolyn A. Maher) and *A Three and One Half-Year Longitudinal Study of Children’s Development of Mathematical Knowledge*, (MDR-9053597, directed by Robert B. Davis and Carolyn A. Maher) and a grant from the New Jersey Department of Higher Education (93-992022-8001, directed by Carolyn A. Maher). This state and federally funded research project provided an innovative in-school mathematics enrichment program to one suburban-rural middle school in central New Jersey (Colts Neck), USA.
NOTE

1 Any opinions, findings or conclusions and recommendations expressed in this volume are those of the authors and do not necessarily reflect the views of the funding agencies.

REFERENCES


CAROLYN A. MAHER AND DINA YANKELEWITZ

1. THE EXPERIMENT

ABSTRACT

This chapter outlines the purpose, methodology, and rationale guiding the interventions described in the book. The intervention setting and tools used in the fourth grade classroom that was the site of the intervention are described. In addition, the forms of reasoning and argumentation that are highlighted throughout the chapters are defined and clarified.

1.1 OVERVIEW

This book describes episodes that are based on video data from an NSF funded classroom study of nine-year old children investigating fraction ideas under conditions that supported investigation, collaboration and argumentation. All of the sessions during the study were videotaped with two or three cameras and the video data, transcripts, and student work were analyzed and stored in an open-source Video Mosaic Repository, VMC (www.videomosaic.org). The chapters that follow are based on analyses of these video data and are accompanied by video narratives (VMCAnalytics), built with the VMC RUanalytic tool to support the descriptions of student reasoning. The accompanying video narratives associated with each chapter show the active student learning and the researcher moves that are made to facilitate student engagement. These analytics form the basis for much of the narrative in each chapter. In addition to referencing the links for analytics in the abstract for each chapter, Appendix C provides a complete list with authors and URL links.

Learning to reason and building justifications for solutions to problems are basic goals in learning in general, and in mathematics learning, in particular. Logical and effective thought have always been defining features in learning and doing mathematics. Schoenfeld (1982) described the discipline of mathematics as a clear and logical analysis and the National Council of Teachers of Mathematics (NCTM) placed reasoning and proof as one of their five process standards from prekindergarten through grade 12 (National Council for Teachers of Mathematics, 2000). Mathematicians and mathematics educators have called for an increased emphasis on reasoning and proof at all levels of learning mathematics, calling for a curriculum that enables the implementation of “a culture of argumentation in the mathematics classroom from the primary grades up all the way
through college” (Ball et al., 2002, p. 907). Earlier research has shown that children, even young children, can build justifications that are “proof like”, that take the form of a mathematical proof such as providing all cases or offering a counterexample to support their solutions to problems (Maher & Martino, 1996a; 1996b). However, in order to foster a learning environment centered on sense making, reasoning and argumentation, certain conditions must be in place. First, children’s ideas need to be heard, shared, and supported. Second, discussion about ideas needs to be encouraged and alternative ideas welcomed and, as appropriate, debated. Third, classroom environments that invite collaboration, questioning, conversation, explanation, and support of ideas and solutions to problems encourage an exchange among students and their challenging of each other’s ideas and solutions. The role of the facilitator was central in promoting an environment that encouraged students’ sharing and questioning. Facilitators were Rutgers University faculty and advanced graduate students, who were prepared by Professors Carolyn A. Maher and Robert B. Davis to maintain the flow of student engagement with the mathematics tasks and problem solving. By setting up opportunities to revisit earlier solutions and arguments, researchers can create conditions for individual activity and working together. Additionally, they can encourage the public sharing of ideas and promote an environment that values sense making, questioning, reasoning, and the building of conceptual understanding that is the foundation for continued learning of mathematics.

Promoting children’s building of conceptual understanding of mathematical ideas is essential before requiring that they apply algorithms and procedures, which often relies on rote memorization, with rules that do not make sense. There is ample evidence that meaningless learning can easily be confused and forgotten (Erlwanger, 1973). If a goal of mathematical instruction is for students to be engaged and confident in their ability to build new knowledge, opportunities must be provided for students to explore, conjecture, build models, and reason. This book provides further evidence that even young children can build arguments that are valid and proof-like. The narratives provide evidence that challenging students to solve open-ended and well-structured tasks and to collaborate and convince others of the validity of their solutions sets the stage for promoting the development of reasoning.

With increased research on effective ways that reasoning and justification can be introduced into classroom learning, we are confident that large-scale changes in learning can occur.

1.2 PERSPECTIVE

The chapters presented here are grounded in an extensive body of research that has been conducted as part of cross-sectional and longitudinal studies of the development of students’ mathematical ideas and ways of reasoning. While the studies occurred in a variety of classroom settings, the work reported here is a component of a three
and one half year cross-sectional study in the suburban/rural community of Colts Neck, New Jersey.

1.3 DATA SOURCE

The classroom sessions for the year-long studies reported here were video and audio taped, with two or three cameras capturing visual data. For each video taping, there was a videographer and an audio technician present. Field notes and student work for each session were collected and preserved. Analyses of particular subsets of the sessions were conducted by doctoral student researchers over the years following the intervention, particularly attending to students’ reasoning patterns (Steencken, 2001; Bulgar, 2002; Reynolds, 2005; Yankelewitz, 2009; Schmeelk, 2010; Horwitz, 2016). The data for these chapters were drawn from transcripts and were cross-checked by at least two researchers. The analyses were reported in a series of Rutgers University doctoral dissertations listed in Appendix B. Student work and video data, with transcripts, are archived in the Rutgers University Video Mosaic Collaborative. The data reported in this volume were collected in the fourth-grade class in the district of Colts Neck in the academic year 1993–1994. During the first half of the year, over a period of four months and twenty-one sessions, the primary focus was to challenge students to construct core fraction concepts such as fraction equivalence, fraction comparison, ordering fractions, operations with fractions and fraction as number.

Professor Carolyn Maher and doctoral student, Amy Martino, led the fourth grade classroom sessions, working with the students of classroom teacher, Joan Phillips, who generously shared her classroom and students with the research team. Principal Judith Landis, who provided the conditions for the study (e.g., parental permission, extended class time) invited parents to join a meeting to learn about the intervention and a classroom session to experience, with their children the problem-solving activities.

1.4 FRACTIONS IN THE CURRICULUM

In the Colts Neck schools, as in many school districts prior to grade 4, the curriculum introduced fractions as operators. In students’ early introduction to fractions, representations were used to show a part of a finite quantity such as a pizza, rectangle, or circle. At the end of the fourth grade year and during the fifth grade year, fraction properties and operations were usually introduced and fractions were represented as numbers, often without reference to quantities. The shift from specific fraction part-whole representations to fraction number representations resulted in obstacles in the learning of many students whose understanding of fraction numbers often reflected their understanding of whole numbers. Even older students, when asked to order unit fractions such as ½, 1/3, ¼ …1/10, incorrectly order the fractions consecutively on a line segment between 0 and 1, attending to the value of the denominators and not
the fraction number values (Maher, Martino, & Davis, 1994; Alston, Davis, Maher, & Martino, 1994). Coordinating student understanding of fraction as operator with fraction as number, a goal of this intervention, is necessary to avoid inappropriate generalizations such as indicated with seventh graders in the Alston et al. paper. Prompted by Davis’ view that building rod models would help students coordinate their operator and number fraction understanding, Researcher Carolyn Maher, assisted by Amy Martino embarked on the teaching experiment described in this volume. The intervention project provided researchers with an opportunity to engage the nine-year old children in problem-solving investigations with fraction ideas prior to their formal classroom instruction of fraction rules, typically taught to students prior to their building an understanding of fraction as number.

1.5 VIDEO NARRATIVES (VMCANALYTICS)

Video data of problem solving of these fourth-grade students revealed students’ process of learning. We have observed students’ cognitive processes in their conjecturing and pursuing validation of their conjectures. As they built rod models to test their ideas, students’ understanding, sense making and reasoning about fraction ideas evolve. Finally, we observed the reconciliation of fraction as operator with fraction as number.

Over the course of the 1993–1994 school year, there were 35 sessions conducted by the researchers. The first 21 sessions focused on fraction ideas. The researchers worked with the students, on average, twice weekly. Each session was approximately 60 to 90 minutes in duration. The 21 sessions that focused on fraction concepts took place between September 20 and December 15, 1993.

One goal of the fraction intervention project was to investigate how students reconcile the ideas of fraction as operator with fraction as number. In the course of their explorations, we observed the representations they use to build an understanding of the concept of the unit as they compared fractions, and built ideas of fraction equivalence. The study explored the ways that students constructed their own strategies for solving fraction problems without previously having been taught traditional algorithms.

1.6 TASKS

The researchers developed open-ended tasks, monitoring carefully the developing ideas of the students and creating new tasks as judgment suggested. When questions were unresolved during a session, the students were invited to revisit the problem during the following session or later, leaving open the question of the correctness of particular solutions for the students, themselves, to resolve. After students worked on each task or group of tasks, they were invited to share their solutions in a whole-class setting, often reconstructing earlier models or building new models of their work to be displayed and shared on an overhead projector. Students were encouraged to justify their solutions, and other class members were asked if they had questions or alternative solutions.
See Appendix A of this volume for a list of the tasks described in the various chapters.

1.7 RESEARCHER ROLE AND SETTING

The primary researchers who conducted the interventions were Rutgers Professor Carolyn A. Maher and Rutgers graduate student Amy M. Martino. The classroom teacher, Joan Phillips, was present during all of the sessions. Occasionally, the principal of the elementary school, Dr. Judith Landis, as well as Rutgers Professor Robert B. Davis, and other members of the research team at Rutgers University, observed the sessions and posed occasional questions as students worked on the tasks. All adults present in the room were instructed not to tell students if their ideas and solutions were correct or incorrect. When students worked in pairs and in small groups, adults were permitted to question students about the work they were doing, elicit explanations, and raise questions to clarify explanations that were offered by the students. The researchers and classroom teacher refrained from influencing the students about the correctness of their solutions. Rather, they attended to eliciting from students justifications for the claims that they were making for the validity of their solutions. The researchers sought to establish the understanding that it was the responsibility of the students to rely on the validity of their reasoning and the reasoning of their classmates. Throughout the study, students were primarily the originators of the ideas that were discussed, challenged, and built. They were expected to provide warrants for the claims that were made.

1.8 THE CLASS

The class consisted of a heterogeneous group of twenty-five students, fourteen girls and eleven boys. Together with the classroom teacher, the researchers often grouped the students in pairs, with the exception of one group of three. Students were encouraged to engage in discussion with other groups of students and to work closely with their partners. Some of the groups changed over the course of the study as an outgrowth of the activity or student interest.

1.9 ORGANIZATION OF CHAPTERS

The reasoning and argumentation reported here occurred during the first 15 of the 21 sessions. Each of the sessions provided opportunities for students to build models with rods to represent their understanding of fraction ideas. Of the six sessions not addressed in this book, two were pencil and paper reviews of concepts developed during earlier sessions; another was a review session during which the children’s parents were present in the classroom; and the final three were sessions involving a real-life exploration involving the division of fractions.
1.10 CUISENAIRE® RODS AS TOOLS

Students had available sets of three-dimensional Cuisenaire® rods, which they used to build models of the fraction ideas that they explored. They were asked to attend to the attribute of length. Each set contains wooden rods of ten different lengths. The rods vary in length by a centimeter difference between each two consecutive rods when ordered from shortest to longest. Each length has a different color name. The shortest rod, white, has length measure of one centimeter; the longest rod, orange, has length measure of ten centimeters and is equivalent in length to a train made with ten of the smallest, one-centimeter rods (Figure 1.1).

Students were directed to attend to the attribute of length when modeling with the rods. A set of transparent overhead rods was also available to project two-dimensional images and was used by the students and the researchers during whole class discussions. In addition to working with individual rods, students made trains of rods by placing two rods end to end and reasoned about the train’s length in comparison with the lengths of other rods or trains of rods (Figure 1.2). Students also had available overhead transparencies to display drawings of their models. In addition, students often used paper and pen to record their solution strategies and explanations.

![Staircase model of Cuisenaire® rods](image)

*Figure 1.1. Staircase model of Cuisenaire® rods*
Students used several forms of reasoning as they presented and justified their arguments. Both direct and indirect forms of reasoning were identified.

Direct reasoning, symbolized by the logical form $p \rightarrow q$, that is read “If $p$, then $q$”, was a common form of reasoning used by students in supporting their ideas. The form is also common in advanced mathematics (Smith, Eggen, & Andre, 2001). The validity of this form of reasoning can be ascribed to the modes ponens rule of logic (which states that, if both $p \rightarrow q$ and $p$ are true, it follows that $q$ is also true).

Indirect reasoning was used to label arguments that followed the form of reasoning by contradiction or by contraposition. Reasoning by contradiction occurred when one attempted to show that a statement is true ($p$). It is first assumed that the denial of the statement is true ($\neg p$), and that assumption is followed to arrive at a contradiction of a second statement ($q$ and $\neg q$). When both this second statement ($q$) and its denial ($\neg q$) are found to be true, it is concluded that $\neg p$ must be false, and that $p$ is therefore true (Smith et al., 2001). An argument by contraposition is defined as one in which a statement, $p \rightarrow q$, is shown to be true by showing the equivalence of $\neg q \rightarrow \neg p$.

In addition to classifying the purpose and structure of arguments exhibited by the students, their forms of reasoning were identified independently. For example, a student might use a direct argument and also use reasoning by cases to organize the argument and tackle each of $n$ cases at a time. However, a student might also use an argument by cases to indirectly show that a statement is true.

1.12 FORMS OF REASONING

Reasoning by cases, also known as the use of an argument by exhaustion, takes the logical form of $p_1 \rightarrow q, p_2 \rightarrow q, \ldots, p_n \rightarrow q$. This form of reasoning organizes the argument by considering a set of finite, distinct cases, and arrives at the same
conclusion after consideration of each case. This form of reasoning requires a systematization of all possibilities into an organized set of cases that can be analyzed separately (Smith et al., 2001).

1.12.2 Reasoning using Upper and Lower Bounds

When reasoning by upper and lower bounds, a student defines the upper and lower boundaries or limits of a class of numbers or mathematical objects. For example, for the set of numbers \{1<x<4\}, the upper bound of the set is 4 and the lower bound is 1, since all the numbers in the set are contained within the two bounds. After these bounds have been defined, the student reasons about the objects between these bounds. This form of reasoning is often used to show that the set that is defined as all objects between the two identified bounds is empty.

1.12.3 Recursive Reasoning

Recursion is “a method of defining functions in which the function being defined is applied within its own definition. The term is also used more generally to describe a process of repeating objects in a self-similar way” (Recursion, as defined in Wikipedia, 2008). A common use of recursive reasoning in advanced mathematics is a proof by mathematical induction. This form of proof shows, for example, that for all natural numbers n, if n is contained in a set, then n + 1 is contained in the set. The proof then concludes that the set is equivalent to all natural numbers.

Recursive reasoning that is used informally in mathematical justification relies on the definition of basic cases and the determination of operations on these basic cases. All operations on any cases in the system can be derived from combinations of the base cases. In this way, the class of objects under study can be built from a few basic cases and rules. Recursive reasoning might be used to show that a calculation is impossible (Recursion, 2008). Also, it might be used to calculate a complex case, using a simpler one and the recursive definition. These recursive functions can then be used to justify a solution of the complex case.

1.12.4 Reasoning using the Generic Example

This form of reasoning is a form of argumentation that can assist students in their journey to formulating valid proofs (Balacheff, 1988; Alibert & Thomas, 1991; Movshovitz-Hadar, 1988; Selden & Selden, 2007). It occurs when a student reasons about the properties of a paradigmatic example that are representative of and can be applied to a larger class of objects in which it is contained and lends insight into a more general truth about that class. The consideration of the general application of these properties in turn verifies the claim made about the particular example.
THE EXPERIMENT

(Rowland, 2002a; 2002b). It is considered a valid form of justification (Balacheff, 1988) that is easily understood by students at all levels, and can be more intuitive than many other forms of proof (Alibert & Thomas, 1991).

1.12.5 Other Forms of Reasoning

1.12.5.1 Generalization. Polya describes generalization as “passing from the consideration of a given set of objects to that of a larger set, containing the given one” (1954, p. 12). This strategy is used to make a statement about a larger set of objects based on observations on a smaller set that is contained in the larger one. Although, this cannot be used to deduce the validity of a statement, it can enable students to discover general properties in mathematics and test their observations to ascertain their validity.

1.12.5.2 Analogical reasoning. Analogical reasoning is useful in mathematics when clarified analogies are used. Two useful forms of clarified analogies are isomorphism and the determination of a similarity of structure or relation between two mathematical propositions, functions, or operations (Polya, 1954). Polya explains that this second form of analogue reasoning is useful “if the relations are governed by the same laws” (p. 29). One common example of this form of reasoning is proportional reasoning, which has a sound mathematical basis and can be used to validate results due to the laws of numbers that are the source of the similarity of relations. Less conventional examples of analogies can be used to explore the properties of partially or completely unrelated mathematical ideas and establish similarity of structure between them.

1.13 SUMMARY

We invite readers to consider the various forms of reasoning and the structures of the argumentation demonstrated in the informal mathematics explorations of the fourth graders as they are illustrated and described in this book. We call your attention to students’ seriousness of purpose, level of engagement, and intensity of their argumentation as they enthusiastically explore fraction ideas.

REFERENCES


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2. ESTABLISHING A MATHEMATICAL COMMUNITY

ABSTRACT

This chapter describes the representations and reasoning used by students to express their understanding of fraction ideas while building solutions to a set of tasks introduced during the first session of the research intervention. In addition to showcasing the foundational skill of building models using Cuisenaire rods to justify or reject the teacher’s or other students’ claims about appropriate number names for the rods, the chapter illustrates the importance of establishing socio-mathematical norms (Yackel & Cobb, 1996) for the acceptability of solutions in the mathematics classroom. In this classroom, norms were immediately established that mandated clearly formed direct justifications that were agreed upon by all members of the community. In addition, the chapter demonstrates the importance of group interaction and ways that students can learn by constructing their own problems in an attempt to challenge their peers. The episodes in this chapter have been described previously (Steencken, 2001; Yankelewitz, 2009), and this chapter describes the reasoning used by the students during the session. This chapter references events from the analytic entitled “Establishing norms and creating a mathematical community,” located in the Video Mosaic Collaborative (http://dx.doi.org/doi:10.7282/T30C4XH9).

2.1 INTRODUCTION

The first session of the fractions intervention took place on September 20, 1993. During this session, researcher Carolyn Maher led the class. Researcher Amy Martino was also present and assisted in questioning students as they worked with their partners. At the start of this session, researcher Carolyn Maher introduced students to Cuisenaire rods (see Figure 2.1) that would be made available for the students to use as tools to explore fraction ideas. The session began by investigating attributes of the rods, providing permanent color names (see Figure 2.1), and emphasizing that number names for the rods would vary, depending on the problem being investigated. In their explorations with the rods, the students ordered them, attending to length, from least to greatest.

The students’ first structured activity included representing the fraction numbers 1/2 and 1/3 using the rods. They were asked to provide justifications for the
models they built. For the most part, these justifications took the form of direct reasoning; however, occasionally, they used indirect reasoning when they presented counterarguments to erroneous claims suggested by the researcher. This phenomenon will be illustrated in the section below.

2.2 INTRODUCTORY TASKS

Task 1: I Claim That the Light Green Rod Is Half as long as the Dark Green Rod. What Do You Think? What Would You Do to Convince Me?

The students made use of the rods to build a solution to the problem (Figure 2.2). Erin, who had not previously used the rods, replied that the claim was true. She justified her solution using direct reasoning by putting two light green rods next to the dark green rod and showing that they were equal in length.

The researcher then asked the students to name the light green rod if the dark green rod was given the number name, one. Kelly indicated that the light green rod would have the number name one half and the rest of the class agreed (VMCAAnalytic Event 1). The students were presented with the next problem:
Task 2: Someone Told Me That the Red Rod Is Half as long as the Yellow Rod. What Do You Think?

When the researcher posed this hypothetical claim, Danielle responded that the statement was not true, and justified her solution by saying, “Two red rods don’t fit. You need to put more.” This explanation was a simplistic version of indirect reasoning, in that it showed why the statement was not true by explaining that length of the two red rods placed together were not equal to the length of the yellow rod (Figure 2.3). The researcher followed this task by posing a similar task that required students to demonstrate why a statement was false using the rods.

Figure 2.2. Modeling one half

Figure 2.3. “Two red rods don’t fit”
Task 3: Someone Told Me That the Purple Rod Is Half as long as the Black Rod. What Do You Think?

The researcher asked the students if the purple rod was half as long as the black rod. Alan and Erik, partners during this session, thought about the problem, and Alan said that it couldn’t be true, because two purple rods were not equal in length to the black rod. Meredith and Sarah, working together, told researcher Amy Martino that the two purple rods were “too large” and that the black rod would need to be longer in order for the statement to be correct. David, when asked during the whole class discussion to respond to the problem, replied similarly, saying: “Two purples are too large.” All four students used indirect reasoning similar to Danielle’s above.

The students were presented with another challenge by the researcher:

Task 4: Someone Told Me That the Red Rod Is One Third as long as the Dark Green Rod. What Do You Think?

Jackie, at the overhead, showed that three red rods are as long as the dark green rod. While she was setting up the rods at the overhead, the researcher called on Michael to explain how he thinks Jackie will justify her solution. Michael explained that by lining up three red rods below the green rod, one could convince others that the statement is true. With this problem, the students extended their use of direct reasoning to think about thirds. Here, norms were set as the researcher asked other students to explain how they thought Jackie would respond, and then asked the rest of the class to confirm that the solution suggested by Jackie was indeed correct.

The researcher then asked the students what number name they would give the red rod if the dark green rod was called one. Sarah showed the researcher that since there were three red rods lined up under the dark green rod, the red rod would be

Figure 2.4. Naming the red rod one third
called one third. The students then exclaimed that they had already built a model to show their solution. When asked to explain what they meant, Beth said that the red rod in the model built for the first part of the task would be called one third (Figure 2.4). Beth elaborated by saying: “Because if you put three on them it makes one whole”. Beth and Sarah used direct reasoning to justify their solution (Event 2).

### 2.3 ESTABLISHING PREMISES

After posing two questions where the solution for different pairs of rods (red and dark green and also light green and blue) was one third, the researcher noted that different rods had been given the number name, one, in the different tasks. She asked the class if number names of the rods could change and if color names of the rods could change. The students answered that the number names could change and eventually came to the consensus that the color names of the rods were permanent. The researcher made sure to establish this important ground rule before bringing up more complex tasks (Event 3).

Then, the researcher established a second important premise that would remain throughout the intervention. She mentioned that she had heard a student suggest that the red rod could be called one if the dark green rod was given the number name, one, and asked the class if this could be possible. Erik said it was not possible and explained that when comparing the red to the dark green the length of the red rod couldn’t be called one (since the dark green was already named one) but when comparing the red rod to another rod it would be possible for it to be given the number name, one (Event 4). The researcher continued by posing the following task.

**Task 5: What Number Name Would I Have to Give to Dark Green If I Wanted Red to be One?**

The problems now shifted students’ thinking from fractions to integers. The researcher asked the class what the dark green rod would be called if the red rod was called one. Erik, working with Alan at his desk, proposed that it be called “three wholes.” He explained to Alan that since three red rods equal a dark green rod, and each red rod was called one, the dark green rod would be called three. Later, when the researcher asked the students to share their solutions in a whole-class discussion, Alan, and then Erik, used this direct argument to justify their solution. Alan said: “Okay, if the red one is considered one, then the green one is a lot bigger. So it would have to be, it would take three whole ones to make another green so it should be considered three wholes.” Erik then repeated this reasoning. David used similar reasoning to justify the solution, saying that if the red rod was called one, “then the green would have to be two more wholes, so that would be three wholes” (Event 5). The researcher presented the following problems.
Task 6a: If I Call Brown One, What Number Name Would I Give to Red?
Task 6b: Now I Want to Call the Red Rod One, What Name Would I Give to the Brown Rod?

This problem again challenged students to consider both integer and fraction names for the rods (Figure 2.5). Danielle explained that if the brown rod was called one, the red rod would be called one fourth. She justified her solution by placing four red rods next to the brown rod. The researcher then asked the students a related question, requiring them to name the brown rod if the red rod was called one. Jacquelyn explained that if the red rod was one, the brown rod would be called four, because four red rods equal a brown rod. Both Danielle and Jacquelyn used direct reasoning to justify their claims (Event 6).

![Figure 2.5. From fractions to integers](image)

Task 7: What Would I Have to Call One If I Want to Name the White Rod One Half?

The researcher challenged the students with one last problem. She asked them what rod would be called one if the white rod was called one half. Laura responded that the red rod would be called one, and Graham justified Laura’s solution directly by saying that this could be shown by placing two white rods next to the red rod and seeing if they were the same length.

2.4 STUDENT-CREATED TASKS

The researcher then asked the students to work with their partners to create tasks that would challenge the class (Event 7). Alan was first to present his challenge. He asked the class, to find the rod that would be called one if the red rod was called one...
fifth. Graham responded that the orange rod would be called one, and explained that five red rods were equal in length to the orange rod (Event 8).

Beth and Mark posed two related challenges. Beth asked the students, “If a [light] green was a whole, what would a blue be?” Erik responded that it would be called “three wholes,” but did not justify his solution. Mark asked the students to name the light green rod if the blue rod was called one. Jacquelyn responded that it would be called one third, and explained that three green rods equaled a blue rod, and so each light green rod would be called one third.

Meredith asked the class to find the rod that would be called one if the purple rod is called one half. Amy replied that the brown rod would be called one, because the purple rod was one half as long as the brown rod, and two purple rods equaled the brown rod.

Next, students worked to create tasks and challenged their partners with them (Event 9). Erik asked Alan, “If a light green was one third, what would be a whole?” (Figure 2.6). Alan responded that the blue rod would be called one, but there was no evidence that he justified his solution. Alan asked Erik to find the rod that would be called one if the white rod was called one fifth. Erik said that the yellow rod would be called one. Researcher Amy Martino asked Erik to justify his solution. Erik began by showing that five white rods equaled a yellow rod in length. He then showed how he also found the solution by using the staircase model he had built on his desk and counting up to the yellow rod, which he called five. He also said, “And I know that that’s half of [the orange rod], and I know that yellow is half of orange, which is ten.” Thus, Erik used three lines of direct reasoning to justify his solution (Event 10).

![Figure 2.6. Erik challenges his partner](image)

Next, Researcher Martino asked Alan and Erik, “If I call this [purple rod] two, what would one look like? Which rod would one be?” Erik replied that the red rod would be called one, and, when asked to justify his solution, explained that the red rod was half as long as the purple rod, and “half of two is one.” He then showed that two red rods equaled the purple rod in length.
2.5 REVIEWING THE REASONING

As students explored these fraction tasks using the rods, direct reasoning was used primarily to justify their solutions. The researchers introduced fraction as number tasks to help students become familiar with the tools and learn to justify any solutions that were presented. In addition to direct reasoning, students used indirect reasoning to counter false claims intentionally introduced by the researcher. These indirect arguments helped students learn that, during these explorations, they were the owners of the mathematics and that they could challenge the facilitators when reason dictated that a claim was incorrect. This is illustrated as the students investigated more complex tasks in future sessions. Chapter 3 includes descriptions of students challenging the claims of others as they thought deeply together about the meaning of one half.

QUESTIONS FOR DISCUSSION OF CHAPTER 2

1. Cuisenaire rods are the physical tools given to the students. What characteristics make this resource a suitable tool for thinking about names for numbers? What obstacles to thinking about fractions might be introduced by the rods?
2. The classroom environment established in this session includes both individual, small group and whole group interactions. What value and problems do you see in this instructional model?
3. What did you notice about the interactions between the facilitator and the students, and among the students that was helpful in establishing a positive learning environment for mathematical reasoning?
4. If you were asked to list the goals of the facilitator for this session, what would you surmise and what evidence would you give for your list? Do you think that she was successful? What evidence can you give for your conclusion?
5. What value, if any, do you see in posing tasks for which there is no correct solution?

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