Teaching to the Math Common Core State Standards
Focus on Grade 5 to Grade 8 and Algebra 1

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This is a methods book for preservice middle level majors and beginning middle school teachers. It takes a very practical approach to learning to teach middle school mathematics in an emerging Age of the Common Core State Standards. The Common Core State Standards in Mathematics (CCSSM) is not meant to be “the” official mathematics curriculum; it was purposefully developed primarily to provide clear learning expectations of mathematics content that are appropriate at every grade level and to help prepare all students to be ready for college and the workplace. A quick glance at the Table of Contents in this book indicates a serious engagement with the recommended mathematics underlying the Grade 5 through Grade 8 and (traditional pathway) Algebra I portions of the CCSSM first, with issues in content-practice assessment, learning, teaching, and classroom management pursued next and in that order.

In this book we explore what it means to teach to the CCSSM within an alignment mindset involving content-practice learning, teaching, and assessment. The Common Core state content standards, which pertain to mathematical knowledge, skills, and applications, have been carefully crafted so that they are teachable, learnable, coherent, fewer, clearer, and higher. The practice standards, which refer to institutionally valued mathematical actions, processes, and habits, have been conceptualized in ways that will hopefully encourage all middle school students to engage with the content standards more deeply than merely acquiring mathematical knowledge by rote and imitation. Thus, in the CCSSM, proficiency in content alone is not sufficient, and so does practice without content, which is limited. Content and practice are both equally important and, thus, must come together in teaching, learning, and assessment in order to support authentic mathematical understanding.

This blended multisourced text is a “getting smart” book. It prepares preservice middle level majors and beginning middle school teachers to work within the realities of accountable pedagogy and to develop a proactive disposition that is capable of supporting all middle school students in order for them to experience growth in mathematical understanding that is necessary for high school and beyond, including future careers.
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CHAPTER 1

DEAR PRESERVICE MIDDLE LEVEL MAJORS AND BEGINNING MIDDLE SCHOOL TEACHERS

An Introduction

This methods book takes a very practical approach to learning to teach middle school mathematics in the Age of the Common Core State Standards (CCSS). The Kindergarten through Grade 12 CCSS in Mathematics (i.e., CCSSM) was officially released on June 2, 2010 with 45 of the 50 US states in agreement to adopt it. Consequently, that action also meant implementing changes in their respective state standards and curriculum in mathematics. The CCSSM is not meant to be “the” official mathematics curriculum; it was purposefully developed primarily to provide clear learning expectations of mathematics content that are appropriate at every grade level and to help prepare all students to be ready for college and the workplace. A quick glance at the Table of Contents in this methods book indicates a serious engagement with the recommended mathematics underlying the Grade 5 through Grade 8 and (traditional pathway) Algebra 1 portions of the CCSSM first, with issues in assessment, learning, teaching, and classroom management pursued next and in that order.

One implication of the CCSSM for you who are in the process of learning to teach the subject involves understanding shared explicit content and practice standards in teaching, learning, and assessing middle school mathematics. The content standards, which pertain to mathematical knowledge, skills, and applications, have been carefully crafted so that they are “teachable and learnable,” that is, smaller than typical in scope, coherent, inch-deep versus mile-wide, focused, and rigorous. According to the Gates Foundation that initially supported the development of the CCSSM, “(t)he new mathematics … standards were built to be teachable and concrete—there are fewer, and they are clearer. And the standards were built on the evidence of what is required for success beyond high school—these standards aim higher.” Further, “coherent” in the CCSSM means to say that the content standards “convey a unified vision of the big ideas and supporting concepts … and reflect a progression of learning that is meaningful and appropriate.” The practice standards, which refer to institutionally valued mathematical actions, processes, and habits, have been conceptualized in ways that will hopefully encourage all middle school students to engage with the content standards more deeply than merely acquiring mathematical knowledge by rote and imitation. An instance of this content-practice relationship involves patterns in numbers. When middle school students are asked
to develop valid algorithms for combining whole numbers (i.e., operations), they should also be provided with an opportunity to engage in mathematical practices such as looking for structures and expressing the regularity of such structures in repeated reasoning.

So, unlike “typical” state standards that basically provide (sound) outlines of mathematical content in different ways, the CCSSM puts premium on this content-practice nexus with instruction orchestrating purposeful learning experiences that will enable all students to learn the expert habits of mathematicians and to struggle productively as they develop mathematical maturity and competence. To perceive the CCSSM merely in terms of redistributing mathematics content that works for all 45 states defeats the purpose in which it was formulated in the first place. That is, the CCSSM is not about covering either the content or practice of school mathematics; it is un/covering content-practice that can support meaningful mathematical acquisition and understanding of concepts, skills, and applications in order to encourage all middle school students to see value in the subject, succeed, progress toward high school, and be prepared for careers that require 21st century skills.

One important consequence of this reconceptualized content-practice approach to learning middle school mathematics involves changing the way students are assessed for mathematical proficiency. Typical content-driven state assessments would have, say, fifth-grade students bubbling in a single correct answer to a multiple-choice item, for instance, choosing the largest sum from several choices of two unit-fraction addends. A CCSSM-driven assessment would have them justifying conditions that make such comparisons possible and valid from a mathematical point of view. Thus, in the CCSSM, proficiency in content alone is not sufficient, and so does practice without content, which is limited. Content and practice are both equally important and, thus, must come together in teaching, learning, and assessment to support growth in students’ mathematical understanding. “One hallmark of mathematical understanding,” as noted in the CCSSM,

is the ability to justify, in a way appropriate to the student’s mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device … and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task … Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

As a document that has been adopted for use in 45 US states, the CCSSM developed appropriate content standards - mathematical knowledge, skills, and applications – at each grade level that all students need to know regardless of their location and context. Such standards are not to be viewed merely as a list of competencies. As noted earlier,
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which is worth reemphasizing, they have been drawn from documented research on learning paths that students tend to pursue, from the initial and informal phase of sense making and adaptive thinking to the more formal and sophisticated phase of necessary mathematical knowledge. Implications of various learning paths then became the basis for designing coherent progressions of mathematical concepts and processes from one grade level to the next. It is this particular constraint that should remind you that they you are teaching not to the common core of mathematics but to the CCSS. Certainly, there are other possible learning paths in, say, understanding integers and rational numbers, which should of course be encouraged and supported. But you must persevere to care and to see to it that the way you teach middle school mathematics supports the original intentions of the CCSSM.

The best available evidence from research and practices across states has also informed the CCSSM content standards. Further, the standards reflect in sufficient terms the content requirements of the highest-performing countries in mathematics. Consequently, you will enter the teaching profession with this available knowledge base that you are expected to know really well and should be able to teach. Certainly, there is always room for interpretation, innovation, and creativity, especially in the implementation phase. However, the CCSSM is unequivocal about the consistent learning expectations of mathematical competence at every grade level, so fidelity is crucial in this early stage of implementation. Own it since you are its very first users. Test it to so that it makes a difference and produces a long-term positive impact on your students. Both the National Governors Association and the Council of Chief State School Officers that sponsored the CCSSM initiative point out that the CCSSM “standards are a common sense first step toward ensuring our children are getting the best possible education no what where they live” and “give educators shared goals and expectations for their students.” And a much greater focus on fewer topics based on coherent learning progressions should make the content-practice doable in every math class. Hence, under the CCSSM framework, you no longer pick and choose. Instead, you teach for content-practice expertise.

With clear content-practice standards in the CCSSM, a common comprehensive assessment became a necessity. As the Gordon Commission has pointed out in relation to the future of education, the troika of assessment, teaching, and learning forms the backbone of a well-conceptualized pedagogy. While the three processes can take place independently, they should co-exist in a mutually determining context. Hence, for you, it means developing a mindset of alignment. In actual practice, in fact, it is impossible in school contexts to conceptualize teaching that has not been informed by any form of assessment of student learning. Further, the basic aim of assessment and teaching is improved learning, no less. Given this mindset of alignment, the Smarter Balanced Assessment (SBA) and the Partnership for Assessment of Readiness for College and Careers (PARCC) emerged, two independent consortia of states that
agreed to develop “next-generation assessments” that are aligned with the CCSSM and ready to be implemented by the 2014-2015 school year. At the core of such high-stakes assessments involves measuring student progress and pathways toward college and career readiness. For you, this institutional expectation of assessment practice also means understanding what is at stake and what is needed to help your students succeed in such high-quality assessments.

Suffice it to say, the realities of the CCSSM, SBA, and PARCC in schools will drive issues of learning, teaching, assessment, and classroom management, including your choice of professional growth and development. Any responsible and thoughtful professional can cope with and work within and around constraints, and this methods book will provide you with much needed support to help you begin to teach to the CCSSM. In past practices, a typical middle school methods course in mathematics would begin with general and domain-specific issues in learning and teaching with content knowledge emerging in the process. The approach that is used in this book assumes the opposite view by situating reflections and issues in learning and teaching middle school mathematics around constraints in content-practice standards and assessment. Like any profession that involves some level of accountability, the manner in which you are expected to teach mathematics in today’s times has to embrace such realities, which also means needing to arm yourself with a proactive disposition that will support all middle school students to succeed in developing solid mathematical understanding necessary for high school and beyond, including the workplace. It is worth noting that the development of the CCSSM, SBA, and PARCC relied heavily on perspectives from various stakeholders such as teachers, school administrators, state leaders and policymakers, experts, educators, researchers, parents, and community groups. For you, this particular collaborative context of the CCSSM, SBA, and PARCC means understanding and appreciating both the efforts and the framing contexts of their emergence. It takes a village for these state-driven initiatives to succeed and it is now your turn to be in the front line of change.

1.1 A BLENDED MULTISOURCED APPROACH TO LEARNING TO TEACH MIDDLE SCHOOL MATHEMATICS

Considering the vast amount of information that is readily available on the internet, including your digital-native disposition and competence toward conducting online searches, this book explores the possibility of a blended multisourced approach to methods of teaching middle school mathematics. Embedded in several sections of this book are activities that will require you to access online information. Becoming acquainted and getting used to this blended form of learning, as a matter of fact, provide good training for the actual work that comes. That is, once you start teaching, you no longer have to plan alone. Furthermore, you do not have to wait for a face-to-face professional development workshop to learn new ideas. You can gain access
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to resources with very minimal cost, and in many cases at no cost to you. What is always needed, however, which applies to any kind of profession, is some kind of training that involves knowing where and how to look for correct and appropriate information. Many sections in this book will require you to access links that provide the best information, including ways to obtain information by relying primarily on email communication. From a practice standpoint, the blended multisourced approach taken in this book symbolizes the collaborative nature of teaching, which you know and seasoned teachers would affirm depend on a repertoire of tools gained by learning from other sources.

For convenience, you are strongly encouraged to access the following free site below, which contains all the links, articles, and reproducible worksheets in a document format that are referenced in various sections in this book. The site was informed and is respectful of copyright rules, so exercise care on matters involving dissemination.

http://commoncoremiddle-schoolmethods.wikispaces.com/

1.2 OVERVIEW OF THE REMAINING CHAPTERS

Chapter 2 introduces you to the eight CCSSM practice standards. Content activities are provided to help you understand the content-practice dimension of teaching, learning, and assessing school mathematics. You will also learn in some detail important psychological and instructional issues surrounding problem solving, reasoning and proof, representations, communications, and connections. Take a peek at Figure 2.1 (p. 14) for an interpretive visual summary of the eight practice standards. Content-practice teaching, learning, and assessment involve establishing appropriate and meaningful correspondences between mathematical content and practices that support the development and emergence of (school) mathematical knowledge.

Chapters 3 through 5 and 7 through 11 deal with content-practice, teaching, and learning issues relevant to the following domains below that comprise the Grade 5 through Grade 8 and (traditional pathway) Algebra 1 CCSSM.

- Real number system and operations (Chapters 3 and 4)
- Ratio and proportional relationships and quantities (Chapter 5)
- Expressions and operations (Chapter 7)
- Equations and inequalities (Chapter 8)
- Functions and models (Chapter 9)
- Measurement and geometry (Chapter 10)
- Data, statistics, and probability (Chapter 11)

Domains consist of clusters or organized groups of related content standards, where each standard defines in explicit terms what students need to understand and be able to do.
Chapter 6 focuses on technology-enhanced tools for teaching and learning middle school mathematics from Grade 5 through Grade 8 and Algebra I. You will explore how to use a graphing calculator, Geogebra, a calculator-based ranger, and apps and other virtual-based manipulatives in ways that support student learning of mathematical concepts and processes. You will also learn how to teach visualization techniques that will help your students attend to what is mathematically relevant in visual-derived representations. “For a mathematician and a teacher,” Raymund Duval writes, “there is no real difference between visual representations and visualization. But for students, there is a considerable gap that most of them are not always able to overcome. They do not see what the teacher sees or believes they will see.” Hence, this chapter will provide you with information that will enable you to orchestrate meaningful and effective instruction (setting up tasks, using guide questions, etc.) with such visual tools.

Chapter 12 addresses different assessment strategies that measure middle school students’ understanding of the CCSSM. It also introduces you to the basic structure and testing requirements of the SBA. You will learn about formative and summative assessments and norm- and criterion-referenced testing, including alternative forms of assessment such as journal writing and unit projects as an extended version of a SBA performance task. Sections that deal with the SBA can be replaced by or discussed together with PARCC, if you find it necessary to do so. In the closing section, you will develop content-practice assessment tasks for practice.

Chapter 13 relies on information drawn from the preceding chapters. You will deal with issues that are relevant to middle school students’ learning of mathematics within the constraints of the CCSSM, SBA, and PARCC. The complex historical relationship between learning theories and the US school mathematics curriculum through the years, including the Math Wars, are also discussed in order to help you understand the negative consequences of holding extreme views on mathematical learning. Further, you will explore in some detail the theories of Piaget and Vygotsky with the intent of helping you understand Fuson’s integrated learning-path developmentally appropriate learning/teaching model, which emphasizes growth in middle school students’ understanding and fluency of mathematics. Fuson’s model offers a middle ground that opposes extreme views and misreadings of constructivist and sociocultural learning in mathematical contexts.

Chapter 14 addresses practical issues relevant to teaching the CCSSM in middle school classrooms. You will learn about different teaching models and write content-practice unit, lesson, and assessment plans.

Chapter 15 focuses on issues relevant to setting up and running an effective mathematics classroom that is conducive for teaching and learning the CCSSM. You will become familiar with issues surrounding student persistence and motivation in mathematics, including ways to design learning and learning environments that foster flexible problem solving and mathematical disposition. You will also be introduced
DEAR PRESERVICE MIDDLE LEVEL MAJORS

to Complex Instruction, an equity-driven approach to group or collaborative learning in the mathematics classroom. General classroom management concerns close the chapter, which deal with how to design and manage optimal learning for all students, eliminate or minimize disruptions, and address potential behavior problems.
In this chapter you will learn about the eight Common Core State Standards for Mathematical Practice (CCSSMP; “practice standards”). Take a moment to access the information from the link below.

http://www.corestandards.org/Math/Practice

The practice standards explicitly articulate how middle school students should engage with the Common Core State Standards for mathematical content (or “content standards”) “as they grow in mathematical maturity and expertise throughout the [middle school] years” (NGACBP & CCSSO, 2012, p. 8). When you teach to the content standards, the practice standards should help inform and guide how you teach, how your students develop mathematical understanding, and what and how you assess for content knowledge. Also, consistent with the overall intent of the Common Core State Standards in Mathematics (CCSSM), the content and practice standards convey shared goals and expectations about the kinds of knowledge, skills, applications, and proficiencies that will help all middle school students succeed in mathematics and be ready for workplace demands. It is worth noting that the practice standards appear to be relevant to other subjects as well, making them powerful, interesting, and useful to middle school students who are learning by the day how making inferences, exercising formal thinking, implementing disciplined reasoning, and forming structures altogether contribute to knowledge structures that are systematic, valid, and exact. Such structures often convey patterns, and mathematics has been characterized as the science of structures. Inference involves going beyond available data, and mathematics is well known for its methods of generalization and abstraction. Both formal thinking and disciplined reasoning demand a conscious application of clearly articulated inferences on the basis of well-defined norms or rules of engagement, and mathematics is the subject that deals with verifying conjectures and proving them in a rigorous and exact manner.

As you pay attention to content-practice relationships in this chapter, you should not lose sight of the assessment aspect of learning, which you will pursue in some detail in Chapter 12 in connection with the Smarter Balanced Assessment. While this chapter focuses on the learning and teaching of a few topics in middle school mathematics in order to demonstrate ways in which mathematical practices support content learning, that is, the unique content-practice dimension of the CCSSM that makes it different from typical state standards in mathematics, it is good to have
an *alignment mindset* at all times. This perspective views effective and meaningful pedagogy to be a fundamental matter of alignment among mathematics curriculum, learning, teaching, and assessment.

Consider the following mathematical problems on parity below, which will help you understand the significance of the eight practice standards. Students learn about the concept of parity – that is, oddness and evenness – of whole numbers as early as second grade. It is a fundamental idea in number concepts that should be familiar to all middle school students. In the elementary grades, students learn to recognize even and odd numbers. Content Activity 1 aims to deepen your and your students’ understanding of parity relationships from basic recognition to justification of general parity relationships.

### 2.1 CONTENT ACTIVITY 1: EXPLORING AND PROVING PARITY

Students in second grade determine the even parity of a set of objects visually by pairing objects and numerically by skip counting by 2s (2.OA.3). This elementary conception is expected to evolve from nonsymbolic to symbolic over time. Work with a pair and do the following tasks below.

- **a.** Use circle chips, unit cubes, or other available manipulatives to model the following even numbers: 2, 4, and 6. Fill in Activity Sheet 2.1.
- **b.** What patterns do you see in Table 2.1? Write a simple multiplicative expression for an even whole number involving variables (5.OA.2, 5.OA.3, and 6.EE.2).
- **c.** Sixth-grade Chase made the following claim about odd numbers: “*When you form pairs of circle chips and always have 1 leftover chip, that number is odd.*” Investigate with manipulatives whether Chase is correct. If he is correct, write an expression that conveys his conjecture in mathematical form (6.EE.2).
- **d.** Is the whole number 0 even or odd? How can you tell for sure?
- **e.** Use manipulatives to help you determine the parity in each case below.
  
  1. Add two or more even numbers.
  2. Add two or more odd numbers.
  3. Subtract two even numbers.
  4. Subtract two odd numbers.
  5. Obtain the sum of an even number and an odd number.
  6. Obtain the difference between an even number and an odd number.
  7. Multiply two or more even numbers.
  8. Multiply two or more odd numbers.
  9. Obtain the product of an even and an odd number.
- **f.** The two situations below involve coin tricks that a sixth-grade teacher shared with her students. Determine how mathematics could be used to solve each problem.
  
  1. Ms. Lamar showed her sixth-grade class a small purse filled with coins.
Ms. Lamar: Okay, I need a volunteer to take some coins from this purse. Jaime volunteered and took a handful of coins.

Ms. Lamar: Jaime, count how many coins you took from the purse. Keep it to yourself or tell your classmates how many you have, but I should not know that information.

Jamie decided to share the information with his classmates.

Ms. Lamar: I will now take some coins from this purse. All I can say is that when I add my coins to yours, it will be even if you have an odd number of coins. It will be odd if you have an even number of coins.

Ms. Lamar added her coins to Jamie’s coins.

Ms. Lamar: Jaime, please count the coins.

Jaime counted and then agreed that the number of coins changed to the opposite. The class was amazed!

Ms. Lamar: I told you. I could change your total number of coins to its opposite!

Activity Sheet 2.1. Parity Table in Compressed Form

<table>
<thead>
<tr>
<th>Even Number</th>
<th>Visual Pairing Process</th>
<th>Sum of Two Equal Addends, or Additive Expression</th>
<th>Product of Two Factors or Multiplicative Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td><img src="image1.png" alt="Image" /></td>
<td>1 + 1</td>
<td>2 \times 1 = 2·1</td>
</tr>
<tr>
<td>4</td>
<td><img src="image2.png" alt="Image" /></td>
<td>2 + 2</td>
<td>2 \times 2 = 2·2</td>
</tr>
<tr>
<td>6</td>
<td><img src="image3.png" alt="Image" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td><img src="image4.png" alt="Image" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td><img src="image5.png" alt="Image" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td><img src="image6.png" alt="Image" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td><img src="image7.png" alt="Image" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ms. Lamar: Okay, I need a volunteer to take some coins from this purse.

Ms. Lamar: Jaime, count how many coins you took from the purse. Keep it to yourself or tell your classmates how many you have, but I should not know that information.

Jamie decided to share the information with his classmates.

Ms. Lamar: I will now take some coins from this purse. All I can say is that when I add my coins to yours, it will be even if you have an odd number of coins. It will be odd if you have an even number of coins.

Ms. Lamar added her coins to Jamie’s coins.

Ms. Lamar: Jaime, please count the coins.

Jaime counted and then agreed that the number of coins changed to the opposite. The class was amazed!

Ms. Lamar: I told you. I could change your total number of coins to its opposite!

How is the trick possible?

2. Ms. Lamar tossed 10 coins on the teacher’s desk. She looked at them quickly and then looked away.
CHAPTER 2

Ms. Lamar: Okay, I need another volunteer to turn over any pair of coins and cover one coin from the pair with his or her hand.

Jennifer volunteered. Ms. Lamar then looked at the exposed coins on the table.

Ms. Lamar: The other coin is a “head.”

Jennifer exposed the other coin and it was, indeed, a “head.”

How is the trick possible?

In the CCSSM, students learn about the number system properties as early as first grade. Check the CCSSM and map the correct content standards with each property listed in Table 2.1.

Table 2.1. Number System Properties Explored in the Elementary CCSSM

<table>
<thead>
<tr>
<th>Number System Property</th>
<th>Applicable CCSSM Content Standards</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative Property of Addition, i.e., $a + b = b + a$</td>
<td></td>
</tr>
<tr>
<td>Commutative Property of Multiplication, i.e., $a \times b = b \times a$</td>
<td></td>
</tr>
<tr>
<td>Associative Property of Addition, i.e., $(a + b) + c = a + (b + c)$</td>
<td></td>
</tr>
<tr>
<td>Associative Property of Multiplication, i.e., $(a \times b) \times c = a \times (b \times c)$</td>
<td></td>
</tr>
<tr>
<td>Distributive Property of Multiplication over Addition, i.e., $a \times (b + c) = (a \times b) + (a \times c)$</td>
<td></td>
</tr>
</tbody>
</table>

In addition to the stated properties in Table 2.2, another basic but important property is closure, which deals with the kinds of numbers you generate when you perform an operation involving at least two numbers in the same set. Closure for addition means that when you choose any two or more numbers from a given set, perform addition on those numbers, and obtain a sum that belongs to the set, then the set is closed under addition. The idea is the same in the case of closure for multiplication. Continue working with your pair to do the following tasks below.

g. Which of the following sets of numbers are closed under addition? multiplication?
   1. Whole numbers
   2. Counting numbers
   3. Proper fractions and mixed numbers in the elementary CCSSM

h. Is the set of counting numbers closed under subtraction? Why or why not? Explain.

i. Is the set of whole numbers closed under division? Why or why not? Explain.
k. Is the set of fractions closed under division? Why or why not? Explain.
l. Sixth-grade Martha used circle chips and inferred the following observation: “If you add two even whole numbers, the result is even. I also think that the set of even numbers is closed under addition.” When she was asked to obtain an expression for the sum of two such numbers, she wrote down the following explanation below.

My first even number is $2a$ and my second even number is $2b$. So, the sum of $2a$ and $2b$ is $2a + 2b$. But I know that $2a + 2b$ is equivalent to $2(a + b)$. So, if $a + b = c$, then the sum of two even numbers is even, which is $2c$.

How was Martha inferring, thinking, and reasoning about her conjecture? Revise Martha’s written solution by supplying reasons where they are needed. The improved solution that contains both statements and reasons is called a mathematical solution to the problem.

If you think that Martha’s method above is valid, go back to tasks (e) and (f) and show that your results are correct by obtaining the appropriate expressions. Provide complete mathematical solutions as well.
m. Is the set of odd numbers closed under addition? If your answer is yes, generate an expression for the sum and provide a mathematical solution. If your answer is no, provide one example to show that it cannot be so. Such an example is called a counterexample. You only need one counterexample to explain why a stated conjecture is not correct.
n. Is the set of even numbers closed under division? If your answer is yes, generate an expression and provide a mathematical solution. If your answer is no, generate a counterexample.
o. Is the set of even numbers closed under subtraction? Explain.
p. Is the set of odd numbers closed under subtraction? Explain.

2.2 THE EIGHT COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

Figure 2.1 shows an interpretive visual model of the eight practice standards. They pertain to institutionally valued mathematical actions, processes, and habits, and have been conceptualized in ways that will hopefully encourage all middle students to engage with the content standards more deeply than merely acquiring mathematical knowledge by rote and imitation. Consider, for instance, the content-practice relationship involving parity in the preceding section. When middle school students are in the process of understanding different valid parity relationships depending on the cases being considered, they should also be provided with an opportunity to
engage in mathematical practices such as looking for structures and expressing the regularity of such structures in repeated reasoning.

Take some time to read each practice standard and the accompanying description. Continue working in pairs and respond to the following tasks below.

a. To what extent did any of the problems in section 2.1 encourage you to:
   1. make sense of the problems and persevere in solving them?
   2. reason abstractly and quantitatively?
3. construct viable arguments and critique the reasoning of others?
4. model with mathematics?
5. use appropriate tools strategically?
6. attend to precision?
7. look for and make use of structures?
8. look for and express regularity in repeated reasoning?

b. Refer to p. 4 of the CCSSM and read the section involving “Understanding mathematics.” To what extent did the content activity encourage you to develop a mathematical understanding of parity and parity relationships?

The practice standards have been derived from two important documents that describe processes and proficiencies that are central to the learning of school mathematics. The National Council of Teachers of Mathematics (NCTM) identifies problem solving, reasoning and proof, communication, representation, and connections to be the most significant processes that all middle school students should learn well. The National Research Council (NRC) characterizes mathematically proficient students as exhibiting mathematical competence in the strands of adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition. Tables 2.2 and 2.3 list the characteristics of the NCTM process standards and the NRC proficiency strands, respectively. Read them carefully and highlight terms and descriptions that you think will require some clarification.

To better appreciate the CCSSM practice standards, explore the following mathematical task, Building a Hexagon Flower Garden Design, from the points of view of the NCTM process standards and the NRC proficiency strands. The task involves generating and analyzing an emerging pattern, which is an appropriate problem solving activity for fifth- through eighth-grade students (5.OA.3, 6.EE.9, 8.F.4, and 8.F.5).

2.3 CONTENT ACTIVITY 2: BUILDING A HEXAGON FLOWER GARDEN DESIGN

Figure 2.2 shows a hexagon flower garden design of size 4 that will be used to cover a flat horizontal walkway between two rooms. A design of size 1 consists of a black hexagon tile that is surrounded by six gray hexagon tiles.

a. Figure 2.2 is a garden design of size 4. How many black and gray hexagon tiles are there?

b. Work with a pair and complete Activity Sheet 2.2 together. What do the four equal signs (=) mean in the context of the activity?

c. Many expert patterners will most likely infer that, on the basis of the assumptions stated for the problem and the data collected in Activity Sheet 2.3, the mathematical relationship appears to model a specific rule. What information relevant to the task and the limited data shown in Activity 2.2 will help them establish that inference?
### Table 2.2. The NCTM Process Standards (NCTM, 2000)

<table>
<thead>
<tr>
<th><strong>Problem Solving (PS)</strong></th>
<th><strong>Reasoning and Proof (RP)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. build new mathematical knowledge through problem solving;</td>
<td>1. recognize reasoning and proof as fundamental aspects of mathematics;</td>
</tr>
<tr>
<td>2. solve problems that arise in mathematics and in other contexts;</td>
<td>2. make and investigate mathematical conjectures;</td>
</tr>
<tr>
<td>3. apply and adapt a variety of appropriate strategies to solve problems;</td>
<td>3. develop and evaluate mathematical arguments and proofs;</td>
</tr>
<tr>
<td>4. monitor and reflect on the process of mathematical problem solving.</td>
<td>4. select and use various types of reasoning and methods of proof.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Communication (CM)</strong></th>
<th><strong>Connections (CN)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. organize and consolidate their mathematical thinking through communication;</td>
<td>1. recognize and use connections among mathematical ideas;</td>
</tr>
<tr>
<td>2. communicate their mathematical thinking coherently and clearly to peers, teachers, and others;</td>
<td>2. understand how mathematical ideas interconnect and build on one another to produce a coherent whole;</td>
</tr>
<tr>
<td>3. analyze and evaluate the mathematical thinking and strategies of others;</td>
<td>3. recognize and apply mathematics in contexts outside of mathematics.</td>
</tr>
<tr>
<td>4. use the language of mathematics to express mathematical ideas precisely.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Representation (RP)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. create and use representations to organize, record, and communicate mathematical ideas;</td>
</tr>
<tr>
<td>2. select, apply, and translate among mathematical representations to solve problems;</td>
</tr>
<tr>
<td>3. use representations to model and interpret physical, social, and mathematical phenomena.</td>
</tr>
</tbody>
</table>

### Table 2.3. The NRC Mathematical Proficiency Strands (NRC, 2001)

<table>
<thead>
<tr>
<th><strong>Conceptual Understanding (CU)</strong></th>
<th><strong>Procedural Fluency (PF)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>comprehension of mathematical concepts, operations, and relations</td>
<td>skill in carrying out procedures flexibly, accurately, efficiently, and appropriately</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Strategic Competence (SC)</strong></th>
<th><strong>Adaptive Reasoning (AR)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>ability to formulate, represent, and solve mathematical problems</td>
<td>capacity for logical thought, reflection, explanation, and justification</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Productive Disposition (PD)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>habitual inclination to see mathematics as sensible, useful, worthwhile, coupled with a belief in diligence and one's own efficacy</td>
</tr>
</tbody>
</table>
Figure 2.2. Hexagon Flower Garden Design of Size 4

Activity Sheet 2.2. Hexagon Flower Garden Design Task Data
d. On the basis of the assumptions noted in (c), three Grade 5 students identified the following features of the pattern below relative to the number of gray hexagon tiles for a design of size 12.

Ava: Keep adding 4, so 2, 2 + 4 = 6, 6 + 4 = 10, 10 + 4 = 14, ..., 46 + 4 = 50.

Bert: Add 12 groups of 4 brown tiles and then add 2 more. It’s like the times table for 4, 4, 8, 12, ..., 40, 44, 48. Then plus 2 you get 50.

Ces: Double 12 groups of 2 brown tiles. Then add 2 more. It’s like the times table for 2 plus 1. So 2, 4, 6, 8, ..., 20, 22, 24, 25. Then you double it and you get 50.

Which student is correct and why? How does each student see his or her pattern relative to the size-12 design?

e. Since students in sixth grade learn to express mathematical relationships involving two quantities with variables and formulas (6.EE.9), the following three Grade 6 students below offered the following formulas below assuming the assumptions in (c).

Dax: My recursive formula is $S = P + 4$. $P$ is the succeeding answer after you add 4 to the preceding answer. So a size-1 tile has 6 gray tiles, that’s $P$ for the succeeding one. So a size-2 tile should have $6 + 4 = 10$ gray tiles. A size-3 should then have $10 + 4 = 14$ gray tiles. And so on.

Eve: My direct formula or rule is $G = 2 + H \times 4$. So a size- tile has $2 + 1 \times 4 = 6$ gray tiles, a size-2 tile has $2 + 2 \times 4 = 10$ gray tiles, etc.

Fey: My direct rule is $T = 6 + (n - 1) \times 4$. So a size-1 tile has 6 gray tiles, a size-2 has $6 + 1 \times 4$ or 10 gray tiles, etc.

Which student is correct and why? How does each student see his or her pattern relative to the size-12 design?

How does a recursive rule or formula differ from a direct rule or formula? What are some advantages and disadvantages of each type of rule or formula? Should sixth-grade students know only one type of rule or formula? Explain.

f. Ava gives Bert 75 brown hexagon tiles in groups of 10. As soon as Bert receives the last tile, he tells her, “It’s odd, so I really can’t use all of the tiles. Do you want 1 tile back?” Ava thinks Bert is not correct. What do you think? If Bert is correct, how does he know so quickly?

Consider your mathematical experience with the Building a Hexagon Flower Garden Design task. Work in groups of 3 and discuss the following questions below. For each group, assign a facilitator, a recorder or scribe, and a reporter. The facilitator
THE COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

needs to make sure that all members of the group are able to share their experiences. The recorder or scribe should see to it that the group report reflects contributions from all members of your group. The reporter is responsible for sharing your group’s views in the follow up whole-classroom discussion. Exercise care and caution as you perform your respective roles so that you all feel respected and are able to make significant contributions in the process of working together.

g. Refer to Table 2.2. To what extent did the above task encourage you to engage in:
   i) at least one of four problem solving actions?
   ii) at least one of four reasoning and proof actions?
   iii) at least one of four communication actions?
   iv) at least one of three connection actions?
   v) at least one of three representational actions?

h. Refer to Table 2.3. This time keep in mind the needs of Grades 5 and 6 students.
   i) What mathematical concepts, operations, relations, and skills do they need in order to successfully construct and justify rules such as the ones the students expressed items (d) and (e)?
   ii) How does having a deep understanding of stable mathematical relationships help them in constructing their formulas?
   iii) What purpose and value do items (d) and (e) serve in their developing understanding of patterns and structures in mathematics?
   iv) How do patterning tasks help them develop productive disposition?

In the next section you will explore the NCTM process standards in some detail, which will help you further deepen your understanding of the issues underlying the practice standards and the work that is required to implement them effectively in your own classrooms.

2.4 PROBLEM SOLVING CONTEXTS IN MIDDLE SCHOOL MATHEMATICS

Problem solving contexts in middle school mathematics are drawn from a variety of situations, as follows:
1. authentic and real situations, where mathematics emerges naturally in the context of students’ everyday experiences;
2. conceptually real situations, where mathematics emerges from well-described scenarios that are rather difficult or almost impossible to be modeled in real time but nonetheless contain sufficient information that will help students perform the relevant actions mentally;
3. simulated (i.e., imitated) situations, where mathematics emerges from situations that are modeled in a technological platform because they either cannot be easily
performed at the physical or concrete level or are meant to conveniently illustrate an instructional objective;
4. context-free mathematical situations, where mathematics emerges from tasks that consist of symbols that students manipulate according to well defined rules or principles.

2.4.1 Content Activity 3: Different Types of Problems in Middle School Mathematics

Access the following link below, which will take you to the Smarter Balanced Assessment released mathematics sample items for grades 5 to 8 students. Work with a pair and answer the questions that follow.

http://sampleitems.smarterbalanced.org/itempreview/sbac/index.htm

a. Solve all the sample items for grades 5 through 8 students first before moving on to the next two tasks below.
b. Use your experiences to help you classify each task item according to the type of problem solving situation it models.
c. Choose five of your favorite sample items. For each item, list the proficiencies that you think are appropriate for students to model and accomplish successfully. Your analysis does not have to reflect all five NRC proficiency strands.

2.5 REPRESENTATIONS IN MIDDLE SCHOOL MATHEMATICS

When you ask middle school students to establish an external representation of a mathematical object, they need to find a way of portraying the object and capturing its essential attributes in some recognizable form. Representations range in type from personal to institutional, informal to formal, and approximate to exact depending on the context in which they arise in mathematical activity. Personal, informal, and approximate representations often reflect individual worldviews, so their forms tend to be idiosyncratic and situated (i.e., they make sense only to the individual). Institutional, formal, and exact representations involve the use of shared inscriptions, rules for combining them, and a well-defined and well-articulated system that links symbols and rules according to the requirements, practices, and traditions of institutions that support their emergence.

Meaningful representations at the middle school level develop in an emergent context beginning with what students already know and how they convey them. There is no other way, in fact, and one goal of teaching involves finding ways to bridge prior and formal representations through negotiation either between you as the teacher and your students or among students in an engaging activity. Figure 2.3 visually captures the sense of emergence in terms of progressions in their representational competence.
2.5.1 Content Activity 4: From Multiplication of Whole Numbers to Direct Proportions to Simple Linear Relationships

Consider the following fifth-grade multiplication task below. Answer the questions that follow.

Watches sell for $175 each. How much would 35 watches cost?

a. Read content standard 5.NBT.5. How are fifth-grade students expected to solve the problem in a multiplicative context? Set up a multiplication expression and obtain the product.
b. The above one-step multiplication problem exemplifies a typical mathematical task that all students solve as early as third grade. Read the cluster of standards 3.OA and develop a version that is appropriate for third-grade students.
c. The above one-step multiplication problem can be also be interpreted as a task that involves ratio and proportional reasoning. Carefully read content standards 6.RP.1, 6.RP.2, and 7.RP.2 to determine the meanings of the following terms: ratio; unit rate; and proportions; and constant of proportionality. Further, read content standards 6.RP.3 and 7.RP.2, which should provide you with different ways of representing the problem in terms of ratio and proportion. Demonstrate
the following representations relative to the problem: (i) tables of equivalent ratios; (ii) tape diagrams; (iii) double number line diagrams; (iv) graph of several ordered pairs of values on a coordinate plane; and (v) linear equation.

d. Solve the following word problem below in different ways. Use the graphic organizer shown in Activity Sheet 2.3 to help you organize your work.

Mrs. Lee, a grocery store owner, bought 3 cantaloupes for $6. How many cantaloupes can Mrs. Lee buy if she has $120?

<table>
<thead>
<tr>
<th>Problem in Words</th>
<th>Table of Equivalent Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tape Diagram</td>
<td>Double Number Line Diagram</td>
</tr>
<tr>
<td>Graph on a Coordinate Plane</td>
<td>Linear Equation</td>
</tr>
</tbody>
</table>

Activity Sheet 2.3. Representing Mathematical Relationships in Different Ways

Personal, approximate, and informal representations involve the use of situated gestures, pictures, and verbal expressions. For example, some fifth-grade students might initially solve the multiplication problem involving the watches through repeated addition of the same addend with very little to no understanding of the conceptual relationship between multiplication of whole numbers and repeated addition. Others might draw a box, label it with the quantity $175, and then perform a series of repeated addition processing. Overall, such representations convey models that are situated (i.e., context-dependent) and either unstructured (i.e., unorganized and naïve) or structured (i.e., sophisticated) in form. Further, they evolve as presymbolic models, that is, they are inductively inferred as a result of seeing repeated actions on similar problems but nothing else beyond that. Students who manage to solve the above problem and other similar examples in a presymbolic mode oftentimes
do not think at the level of deductive rules that apply to all problems of the same type. The presence of a real-life context also characterizes the presymbolic and, thus, nonsymbolic, dimension of such representations. Your goal as the teacher is to help them develop and manipulate symbolic representations. Symbolic representations in (school) mathematics convey decontextualized mathematical relationships that operate and make sense at the level of structures and rules alone.

Classroom representations mark the beginning of progressive formalization. Students’ representations in this phase are purposeful and intentional as a result of their intense and shared interactions with you and others learners. Through meaningful discussions they learn to use tape diagrams and ratio tables, which replace actions of extensive and (un)structured listing or repeated addition of the same addend. Classroom representations achieve their meaningful progressively formal state when middle school students begin to coordinate the following two nonsymbolic aspects of their representations below.

1. **Analytic Condition**: Knowing how to process problems deductively by applying the appropriate rules;
2. **Abstract Entity Condition**: Using formal symbols to translate problems in external form.

In this particular characterization of classroom representation, middle school students think in terms of rules. In the presymbolic phase, specific instances over rules dominate representational action. In the nonsymbolic phase, the reverse is evident. However, such representations remain grounded in the nonsymbolic phase because they are still strongly linked to the context of their emergence. For example, in the case of the above multiplication task involving watches, the numbers 175 and 35 are classified as arithmetical quantities, that is, numerals with units. The number sentence

\[ 35 \times 175 = 6125 \]

actually means

\[ 35 \text{ watches } \times \frac{\$175}{1 \text{ watch}} = \$6125 \]

or,

\[ 35 \text{ watches } \times \$175/\text{watch} = \$6,125. \]

Another interesting characteristic of classroom representations is the condensed or contracted nature of the forms involving alphanumeric expressions that replace the oral, verbal, and pictorial descriptions that are evident in the presymbolic phase.

Institutional representations convey formal and shared representations at the community level. Since school mathematics content reflect the values, traditions, and practices of the entire mathematics community, middle school students need to acquire appropriate and intentional ways of representing objects and relationships in order to participate meaningfully in the larger community. Institutional representations evolve into symbolic models as soon as students begin to decontextualize problems, that is,
by shifting their attention from the contexts to the processing of the abstract entities. 
Think about the above multiplication tasks together. Such problems evolve in purpose 
over time in the middle school mathematics curriculum. In third grade, students 
process them by using equal groups, arrays, drawings, or simple equations with 
appropriate symbols for the unknown numbers to represent the problems (3.OA3). 
In fourth grade, they represent them in the context of multiplicative comparisons and 
multiples (4.OA.2 and 4.OA.3) and with larger numbers (4.NBT.5). In fifth grade, 
they either focus on writing simple expressions that record the relevant calculations 
without having to solve the problems (5.OA.2). Or, if they need to solve them, they 
are expected to use the standard algorithm and deal with factors involving multidigit 
numbers (5.NBT.5). From sixth- through eighth-grade, they learn to see the structures 
of such problems in terms of ratios (6.RP.1 to 6.RP.3), proportions (7.RP.1 to 7.RP.3), 
and simple linear function or power relation of the type \( y = mx \) (7.RP.2c, 7.RP.2d, 
8.EE.5, 8.EF.1 to 8.EF.2, F-IF.1, F-IF.2, and F-IF.1 to F-IF.7a). 
Moving on to a different aspect of representation, some mathematical objects are 
known to generate multiple representations, that is, they can be expressed in several 
different ways. For example, Activity Sheets 2.2 and 2.3 illustrate different ways of 
representing patterns and proportional relationships, respectively. Work with a pair 
and accomplish the following activity below.
e. Think about all the mathematical objects you know that have multiple 
representations. Many middle school teachers tend to teach all the relevant 
representations involving the same object at the same time. Do you agree? Discuss 
possible concerns with this particular teaching practice.

2.6 CONNECTIONS IN MIDDLE SCHOOL MATHEMATICS

One important aspect of your job involves helping middle school students establish 
connections among mathematical ideas. If you think of connections in terms of 
relations, then mathematical ideas are in fact linked in at least two different ways 
depending on the context of activity, as follows:

• **Representational connections** involve matters that pertain to equivalence, 
  which means that two or more representations convey the same idea despite the 
  superficial appearance of looking different. Consider, for example, the multiple 
  representations for the same pattern shown in Activity Sheet 2.2. The table, 
  ordered pairs, graph, and rule are all equivalent representations of the pattern 
  shown in Figure 2.2. They hold the same assumptions about the pattern, which 
  support their equivalence. Also, take note of the four equal symbols shown in 
  Activity Sheet 2.2, which convey the sense that equal means “is the same as.”

• **Topical connections** involve linking two or more separate topics either from the 
  same subject (e.g., multiplication of whole numbers and ratio and proportion) or 
  from different subjects (e.g., the Pythagorean Theorem in geometry and irrational 
  numbers in algebra).
Meaningful connections should facilitate \textit{transfer}, which is a core process that supports thinking and learning about structures. Transfer involves applying what one has learned in one situation to another situation. But that is usually easier said than done. Transfer is a subjective experience for individual learners, meaning to say that they need to establish the connections or relationships between two or more situations, representations, or topics themselves. Certainly, the subjective process can be smoothly facilitated by providing them with tools at the appropriate time that will enable them to see those connections as similar in some way. In other words, connections among representations and topics do not transfer on their own. Learners need to construct them with appropriate resources at their disposal. Such resources can be concrete (e.g., using manipulatives) or interventional (e.g., using purposeful guide questions) to help them accomplish successful transfer. The following activity exemplifies topical connections. Work with a pair and answer the questions that follow.

2.6.1 \textbf{Content Activity 5: The Pythagorean Theorem and Irrational Numbers}

Middle school students first learn about the Pythagorean Theorem in eighth grade. Content standard \textbf{8.G.6} expects them to provide a mathematical explanation for the correctness and validity of the theorem for any right triangle. Furthermore, content standard \textbf{8.G.7} expects them to apply the theorem in situations that involve determining an unknown side length in a right triangle in both contextual and decontextualized situations. Interestingly enough, they also learn about radicals (\textbf{8.EE.1} and \textbf{8.EE.2}), especially irrational numbers (\textbf{8.NS.1} and \textbf{8.NS.2}), for the first time in eighth grade. In this section, you will learn how the Pythagorean Theorem is topically related to the mathematical way of establishing the existence of square roots (i.e., radicals with an index of 2). Work with a group of 3 students and do the following tasks below.

\textbf{a.} Discuss the following two prerequisite concepts before moving on to the next step: (i) Make sure that you remember how to read and write expressions involving whole-number exponents (\textbf{6.EE.1}). Students are exposed to exponents for the first time in the context of representing the decimal number system in terms of powers of 10 (\textbf{5.NBT.2}); (ii) Students also learn about the area of a square with a side length of \textit{s} units in third grade (\textbf{3.MD.7b}). Explain why its area A is \textit{s}^2 square units.

\textbf{b.} The diagrams in Figure 2.4 provide two visual ways to verify the correctness of the Pythagorean Theorem. The theorem states that any right triangle with side lengths \textit{a}, \textit{b}, and \textit{c} units, where \textit{c} is the hypotenuse or the longest side of the triangle, will obey the mathematical relationship \(a^2 + b^2 = c^2\).

For the diagram on the left, obtain the areas of the two squares. Then use a pair of scissors to cut out figures 2 through 5 along the dotted segments. Next, use all five figures to form another square. How is the new square related to the length of the hypotenuse of the right triangle? Verify that it is, indeed, the case that \(a^2 + b^2 = c^2\).
The diagram on the right starts with four copies of the same right triangle. Move the triangles around so that you form the square shown. Then calculate the areas of the larger square and the individual components of the smaller squares. How are they related to each other? Verify that it is, indeed, the case that $a^2 + b^2 = c^2$.

\[ a^2 + b^2 = c^2 \]

Figure 2.4. Exploring the Pythagorean Theorem in Two Ways

Find the length of the hypotenuse in each right triangle below.

i. \[ 3^2 + 4^2 = (\text{?})^2 \]  
ii. \[ 5^2 + 12^2 = (\text{?})^2 \]

c. Figure 2.5 shows a square with a side length of 1 unit. Cutting the square along a diagonal yields a right triangle. Calculate the length of the hypotenuse. Can you find a whole number that will complete the equation shown in Figure 2.5? How about a fraction? Would a decimal number work? Try to see what happens when you fill in the blank with the exponential expression $2^{\frac{1}{2}}$ or, in radical form, $\sqrt{2}$ (“square root of 2,” could also be written as $\frac{\sqrt{2}}{2}$).

\[ 1^2 + 1^2 = (\text{?})^2 \]

Figure 2.5. Exploring $\sqrt{2}$
d. Since $\sqrt{2}$ is not a whole number, it is reasonable to ask if it is a rational number (i.e., a quotient of two integers). Investigate what happens when you assume that it is a rational number by following the argument below.

If $\sqrt{2}$ is a rational number, then $\sqrt{2} = \frac{a}{b}$, where $a$ and $b$ are integers that do not share any common factor and $b \neq 0$ (why?).

Squaring each side of the equation, which is permitted, yields the equivalent equation $2 = \frac{a^2}{b^2}$, or $a^2 = 2b^2$. Why?

Now, observe that $a^2$ is even. Why?
Consequently, $a$ must be even and $a^2$ must have a factor of 4. Why?
Consequently, too, both $b^2$ and $b$ must be even. Why?
Since $a$ and $b$ are both even, this contradicts an assumption. Which one?
Thus, $\sqrt{2}$ cannot be a rational number. It is, in fact, an irrational number.
How does content standard 8.NS.1 describe an irrational number?
What does content standard 8.NS.2 say about the approximate value of $\sqrt{2}$?
How can you use the number line below and the right triangle in Figure 2.5 to visually verify the approximate value?

```
   0  1  2  3
Figure 2.5. Number Line Diagram
```

e. Use your knowledge of the Pythagorean Theorem to generate five irrational numbers. Approximate their values visually and numerically.

2.7 REASONING AND PROOF IN MIDDLE SCHOOL MATHEMATICS

Knowing how to reason and prove mathematically is just as equally important as knowing how to process problems deductively by applying the appropriate rules. Work through the following activity below with a partner.

a. There exists an infinite number of irrational numbers. Show that $\sqrt{5}$ exists by finding a right triangle whose side lengths yields a hypotenuse of $\sqrt{5}$ units. Then use the argument structure in section 2.6.1 item (d) to develop an explanation that $\sqrt{5}$ is an irrational number. Also, obtain an approximate value of $\sqrt{5}$ visually and numerically.
CHAPTER 2

Reasoning and proof are fundamentally tied to beliefs. When students reason, they convey beliefs that are usually based on inferential norms that often map to their level of experience. Certainly, those beliefs evolve over time depending on how well they are able to coordinate factors relevant to perceptual and conceptual competence and representational ability.

- **Perceptual competence** involves making inferences on the basis of what one sees at the moment.
- **Conceptual competence** involves making analytical inferences on the basis of one’s ability to interpret valid mathematical relationships, properties, features, or attributes of objects beyond the limits of perceptual competence.
- **Representational ability** involves employing gestures, drawing pictures and diagrams, and using abstract entities (numeric forms, variables, and combinations) that enable one to externalize concepts and percepts.

Reasoning and proof activities help deepen middle school students’ developing mathematical understanding. Activities that encourage them to generate propositions or conjectures and claims or provide explanations that justify their truth values (e.g.: why they are either true or false; how they are true or false; under what conditions they are either true or false) help them establish certainty and necessity in their developing mathematical knowledge. Many of them, of course, tend to reason through examples that are intended to empirically demonstrate certainty at their level of experience.

Further, the examples they use and the repeated actions that come with them convey how they sense and capture the necessity of an interpreted structure. These kinds of explanations and actions should be encouraged and supported.

When you teach middle school students to reason mathematically, you should aim to help them achieve competence in three different kinds of explanations, as follows:

- **Abductive reasoning**, which involves forming conjectures or claims;
- **Inductive reasoning**, which involves performing repeated actions as a way of demonstrating the validity of conjectures or claims;
- **Reasoning that emerges from a deductive argument**, which arises from abductive and inductive reasoning together.

For example, consider the following steps that demonstrate the parity of the sum of any pair of even whole numbers.

I. 2 + 2 = 4. The sum is even.
II. 4 + 6 = 10. The sum is even.
III. 2 + 8 = 10. The sum is even.
IV. 6 + 6 = 12. The sum is even.
V. Two even addends yield a sum that is even.
VI. So 26 + 12, which equals 38, should be even.

Each claim under steps I, II, III, and IV closes with an abduction, that is, a guess or conjecture. Steps I through IV together support step V, which illustrate a statement
that emerges from inductive reasoning. Step VI involves reasoning that emerges from
the preceding five steps; it exemplifies what is called a *deductively-closed argument*
that is empirically supported by both abduction and induction. When you ask middle
school students to reason about any mathematical object or relationship, you should
aim for deductive closure at all times, which is a fundamental characteristic of the
analytical condition of mathematical representations. Work with a pair and deal with
the following two questions below.

g. What happens when middle school students are not provided with
   sufficient opportunities to engage in abductive and inductive reasoning?
h. What happens when middle school students are able to engage in abductive and
   inductive reasoning but unable to achieve deductive closure?

2.8 COMMUNICATION IN MIDDLE SCHOOL MATHEMATICS

When you ask middle school students to communicate in the mathematics classroom, it
means that you want them to pay serious attention to forming, using, reasoning with, and
connecting various forms of representations. Communication involves externalizing an
idea in some recognizable form for personal and public access. Personal access is a form
of intra-communication that helps them monitor what and how they are thinking when
they are in the process of conceptualizing or problem solving. However, meaningful
communication takes place in a public domain, where individuals have access to each
other’s representations. Hence, public access is a form of inter-communication that
helps them monitor and learn from each other with the goal of developing shared
intent, knowledge, and practices over time. Meaningful communication in public also
targets the development of more refined and sophisticated forms of representation and
reasoning, including the relevant connections that make them interesting. Furthermore,
because personal communication of mathematical representations among middle
school students is still in a developing phase, which can be inexact or situated and
organized or unorganized, their experiences in public communication provide them
with structured opportunities that enable them to negotiate and bridge conceptual and
representational gaps between personal and formal or institutional versions. Thus, this
purposeful public communication in mathematical activity supports the emergence
and full development of exact and valid forms of representations.

Do not be surprised if the initial forms of public communication among
middle school students appear to be predominantly verbal, gestural, pictorial, or
combinations of at least two such forms. Also, depending on grade level, some of
them will need manipulatives and other concrete tools to help them communicate.
Consider the first task in Content Activity 4, which involves determining the total
cost of purchasing 35 watches that sell for $175 each. Some students who are
proficient in English (or in an official language) are already capable of explaining
their thinking in a mathematical way. However, there are students such as English
learners and those who are not ready to share their personal thoughts in public who
will employ gestures or draw pictures as their preferred ways of communication. Verbal, gestural, and pictorial means of public communication should be supported, of course. However, they are good tools to think but only when students are able to extract from public discourse the analytic conditions of representations across specific instances, examples, or tasks. Knowing such conditions is, in fact, the essence of academic language acquisition in a public forum. No less.

Writing solutions is the most significant form of public communication in mathematics. Among middle school students, it involves drawing pictures and diagrams, setting up number expressions and number sentences (i.e., equations and inequalities), graphing mathematical relationships, constructing tables of values, etc. Unlike verbal and gesture-driven solutions as forms of communication, however, having them write their solutions involves asking them to externalize in paper their understanding of “all aspects relevant to language use, from vocabulary and syntax to the development and organization of ideas” (NGSS/CCSS, 2011, p. 19). Further, “a key purpose of writing is to communicate clearly to an external, sometimes unfamiliar audience” and “begin to adapt the form and content of their writing to accomplish a particular task and purpose” (ibid, p. 18). While these perspectives on writing have been taken from the appropriate sections of the Common Core State Standards for English Language Arts and Literacy in History/Social Studies, Science, and Technical Subjects, they also apply to writing solutions in school mathematics.

When you ask middle school students to write solutions to mathematical problems, your most significant task is to help them coordinate between the following actions below, which they need to clearly manifest in order to develop a good writing structure.

- **Processing action**, which involves drawing pictures and diagrams that show how they perceive and conceptualize relationships in a given problem.
- **Translating action**, which involves converting the relevant processing action in correct mathematical form.

Asking them to draw pictures as an initial step can help them make the transition to the formal phase of writing solutions. Further, drawn pictures can evolve into (schematic) diagrams over time with increased conceptual competence. Pictures are iconic forms of representation, meaning they remain faithful to the depicted objects. For example, a word problem that involves apples and oranges in a ratio problem might yield a written solution with drawn apples and oranges. Some drawn pictures might also contain features that are irrelevant to a problem. However, with appropriate scaffolding, middle school students can be taught to transform those pictures into diagrams.

Diagrams are either indexical or symbolic forms of representation. Indexical forms are figures that emerge out of association, while symbolic forms are figures that emerge from shared rules of discourse. For example, the square root form shown in section 2.6.1 item (c) involving a radical sign is not symbolic but indexical since the form reminds students to perform a square root process and nothing else. Variables such as X, Y, and Z are symbols that the mathematics community use for
THE COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

certain purposes. Indexical and symbolic diagrams can be mere skeletal shapes of figures or placeholder figures that do not need to resemble the original source.

Middle school students are naturally drawn to process mathematical problems due to the interesting contexts in which they arise or are framed. However, many of them often find it difficult to convert their processing actions in correct and acceptable mathematical form. Explore the following activity below with a pair. Pay attention to potential issues in processing and converting representations in a problem-solving activity for fifth-grade students.

2.8.1 Content Activity 6: Solving Multiplication Problems with Tape Diagrams

By the end of fifth grade, students are expected to be able to solve real world problems involving multiplication of fractions either by using visual fraction models or equations (5.NF.6). Visual fraction models are instances of tape diagrams, which the CCSSM describes as drawings “that look like segments of tape, used to illustrate number relationships.” Tape diagrams are otherwise known as “strip diagrams, bar models, fraction strips, or length models.” Consider the following problem below.

Two sisters sold caramel apples in a school fair. Maria’s earnings were \( \frac{3}{5} \) of Nini’s earnings. Together, they earned $520. How much did Nini earn?

a. Figure 2.6 provides a beginning tape diagram processing of the problem. How does the diagram help you solve the problem?

b. Convert the processing actions you performed in (a) in mathematical form. That is, write a complete mathematical solution to the given problem.

Three fifth-grade students, Mark, June, and Rhianna, compared their weekly savings in the following manner:

Mark: The value of my weekly savings is \( \frac{4}{6} \) of June’s weekly savings.

Rhianna: In my case my total value is only \( \frac{3}{4} \) of your weekly savings.

If June’s weekly savings is $30, how much does Rhianna save in a week?
2.9 DOING MATHEMATICS WITH AN EYE ON THE CONTENT-PRACTICE STANDARDS OF THE CCSSM

Hopefully the preceding sections have given you sufficient and meaningful insights and experiences that encouraged you to reflect on how you might begin to structure content teaching and learning with the practice standards in mind. You will learn more about the relationships between content and practice standards in the next eight chapters that focus on the core domains in the CCSSM middle school mathematics curriculum. *Domains* consist of *clusters* – organized groups – of related standards, where each *standard* defines in explicit terms what every middle school student needs to understand and be able to do. The eleven middle school mathematics domains up to Algebra 1 in the CCSSM are as follows:

- Operations and algebraic thinking in Grade 5;
- Numbers and operations in base 10 in Grade 5;
- Numbers and operations involving fractions in Grade 5;
- Ratios and proportional relationships from Grade 6 to Grade 7;
- Number system from Grade 6 to Grade 8 and Algebra 1;
- Quantities in Algebra 1;
- Expressions and equations (and inequalities) from Grade 5 to Grade 8 and Algebra 1;
- Functions in Grade 8 and Algebra 1;
- Geometry from Grade 5 to Grade 8;
- Measurement and data in Grade 5;
- Statistics and probability from Grade 6 to Grade 8 and Algebra 1

An example of a cluster of three standards under the CCSSM Grade 5 Operations and Algebraic Thinking domain is shown in Figure 2.7. Write and interpret numerical expressions.

<table>
<thead>
<tr>
<th>Grade 5 Operations and Algebraic Thinking</th>
<th>5.OA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write and interpret numerical expressions.</td>
<td></td>
</tr>
<tr>
<td>1. Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.</td>
<td></td>
</tr>
<tr>
<td>2. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. For example, express the calculation “add 8 and 7, then multiply by 2” as 2 × (8 + 7). Recognize that 3 × (18032 + 921) is three times as large as 18032 + 921, without having to calculate the indicated sum or product.</td>
<td></td>
</tr>
<tr>
<td>Analyze patterns and relationships.</td>
<td></td>
</tr>
<tr>
<td>3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. For example, given the rule “Add 3” and the starting number 0, and given the rule “Add 6” and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 2.7. CCSSM Format*
Note that Figure 2.1 is an interpretive visual model of the CCSSM practice standards. While the CCSSM considers all practices as essential in supporting growth in mathematical understanding, the perspective that is assumed in this book embraces a stronger position. The solid triangle in Figure 2.1 conveys three core mathematical practices that all middle school students need to demonstrate each time they learn mathematical content. Having a positive disposition toward mathematical problems and problem solving is central. Knowing which tools to use, formal or informal, and when to use them further support and strengthen such disposition. Also, given the symbolic nature of school mathematical content, which is both formal and exact, attending to precision at all times introduces an algebraic disposition toward tool use in which case emerging and meaningful symbols convey expressions that are certain and necessary in the long term. Any of the remaining practice standards can then be coupled with the solid triangle depending on the goals of mathematical activity.

Work with a pair and accomplish the following two activities below.

First, access the following link which describes in detail the eight practice standards: http://www.corestandards.org/Math/Practice. For each practice standard, develop a checklist of specific actions that is appropriate for middle school students. Some statements under each practice standard may not be appropriate for middle school students, so consult with each other and decide which ones matter. You may refer to Table 2.4 for a sample of a beginning checklist structure for Practice Standard 1.

Second, in this chapter you explored six content activities. Use the checklist in Table 2.5 to determine practice standards that you can use to help your students learn each content activity. When you teach with a content-practice perspective, you (and your colleagues) decide which practice standards to emphasize over others, but you should also see to it that such choices are aligned with your teaching, learning, and assessment plans.

<table>
<thead>
<tr>
<th>Practice Standard 1: Make sense of problems and persevere in solving them.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Provide middle school students with every opportunity to –</td>
</tr>
<tr>
<td>1.1 start by explaining to themselves the meaning of a problem and looking for entry points to its solution.</td>
</tr>
<tr>
<td>1.2 analyze givens, constraints, relationships, and goals.</td>
</tr>
<tr>
<td>1.3</td>
</tr>
<tr>
<td>1.4</td>
</tr>
<tr>
<td>1.5</td>
</tr>
<tr>
<td>etc.</td>
</tr>
<tr>
<td>Content Activity</td>
</tr>
<tr>
<td>------------------------------------------------------</td>
</tr>
<tr>
<td>Exploring and Proving Parity</td>
</tr>
<tr>
<td>Building a Hexagon Flower Garden Design</td>
</tr>
<tr>
<td>Different Types of Problems in Middle School Mathematics</td>
</tr>
<tr>
<td>From Multiplication of Whole Numbers to Direct Proportions to Simple Linear Relationships</td>
</tr>
<tr>
<td>The Pythagorean Theorem and Irrational Numbers</td>
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<tr>
<td>Solving Multiplication Problems with Tape Diagrams</td>
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</tbody>
</table>