This book shows how geometry can be learned by starting with real world problems which are solved by intuition, common sense reasoning and experiments. Gradually the more formal demands of mathematical proofs get their proper place and make it possible to explore new applications. This process helps students to feel the need for precise definitions and procedures, to contribute to the construction of an axiomatic system, and to experience the power of systematic reasoning.

The course is designed for students in a Nature & Technology strand which prepares for studying the sciences or technology at university level. Its goal was basically to reintroduce ‘proof’ in a meaningful way in the late 1990s Dutch secondary education curriculum. Following the educational view of the Freudenthal Institute this is not done by stating Euclid’s axioms on page one, but rather a starting point is chosen in students’ intuitions and tentative solutions of problems that are experienced as real and relevant.

The photograph on the cover shows students exploring one of the problems from the midpart of the course in the computertab.
Geometry with Applications and Proofs
Advanced Geometry for Senior High School,
Student Text and Background Information

Aad Goddijn
Freudenthal Institute for Science and Mathematics Education,
Utrecht, The Netherlands

Martin Kindt
Freudenthal Institute for Science and Mathematics Education,
Utrecht, The Netherlands

and

Wolfgang Reuter
Schoter Scholengemeenschap, Haarlem, The Netherlands

SENSE PUBLISHERS
ROTTERDAM / BOSTON / TAIPEI
Contents

Introduction 1
Geometry between application and proof, a general introduction 1
Geometry, classical topics and new applications 9
Given: circle with butterfly or: how do you learn proving? 23

Part I: Distances, edges and domains 41
Chapter 1: Voronoi diagrams 43
Chapter 2: Reasoning with distances 61
Chapter 3: Computer practical Voronoi diagrams 81
Chapter 4: A special quadrilateral 93
Chapter 5: Exploring isodistance lines 107
Chapter 6: Shortest paths 129
Example solutions 141
Worksheets part I 173

Part II: Thinking in circles and lines 183
Chapter 1: Using what you know 185
Chapter 2: The circle scrutinized 195
Chapter 3: Finding proofs 207
Chapter 4: Conjectures on screen 219
Chapter 5: Proving conjectures 231
Clues for chapter 3 and 5 241

Part III: Conflict lines and reflections 251
Preface 253
Chapter 1: Edge and conflict 255
Chapter 2: Parabola, ellipse and hyperbola 271
Chapter 3: Analytic geometry 301
Chapter 4: Conic sections 331

Sources of some of the illustrations 341
Geometry between application and proof,  
a general introduction

Aad Goddijn

About this book

The main parts of this book (I–III) make up a course in geometry for secondary education, specially designed for students in the Nature & Technology strand, which prepares students in their last three years of secondary education for studying the sciences or technology at university level. The course was used in an experimental setting by schools in the so-called Profi project. Its goal was basically to reintroduce “proof” in a substantial way in the Dutch secondary education curriculum in the 1990s. This book contains the greater part of the geometry course. To restrict the size of the book, some parts were not included; we restricted ourselves to chapters, subjects and tasks which characterise the specific approach to geometry well. Readers and educators who really love mathematics and teaching may immediately try their hand at part I–III, but some background information about the materials may be useful nonetheless. That is why we include this short general introduction, an explanation about the combination of application and proof in the educational approach and some stories from the classroom about learning to prove.

Geometry in Dutch education

Geometry in the first years of Dutch secondary education is often strongly related to realistic experiences, and is in a way highly intuition-based. Exploring spatial objects and shapes, relating different types of images of objects and situations, calculating with proportions on similar figures, some Pythagoras, computation with angles, most of the time in concrete situations, are what the focus is on. Discussion is elicited, but remains in general situation-dependent and the abstraction level is quite low. Understanding the physical world with the help of some basic mathematical tools is the main goal. It is a geometry for daily life, also
preparing well (or at least not too badly) for vocational schooling and practical jobs.

This junior high school geometry curriculum was developed in the early nineties, as a counterbalance for the then prevailing boring geometry tasks. Currently there is a tendency to move part of this intuition-based (and intuition-stimulating) type of geometry to its proper place in primary education; an outline of possible goals and learning-teaching trajectories has been published recently by the Freudenthal Institute. An English translation is available.¹

In the same period many teachers, especially at schools of the traditional ‘gymnasium’-type (similar to ‘grammar schools’ in the UK, streaming for university), liked to include more and more thought-provoking and proof-related problems in their geometry teaching, in the certainty that their students can do more intellectually than the curriculum seems to allow.

The present geometry course for upper secondary education in this book fits very well in this environment: in the course, a good orientation base of intuitive insight in geometry will help in becoming familiar with the more formal demands of mathematical proofs. We tried to link the intuitive and formal approaches without mixing them up, including clarifying for the students what the differences are. In this respect the first chapter (Geometry, classical topics & new applications, by Martin Kindt) is instructive. Important moments in the course around this theme can be found in chapter 2 of part I, Reasoning with distances, and chapter 1 of part II, Using what you Know.

**Mathematical contents of the course**

Part II (Thinking in circles and lines) is the closest approach the course offers to The Elements of Euclid; the title can be read as a reference to the well-known ‘ruler and compass’. But no attempt has been made to cover the first six books of The Elements, where the traditional secondary education geometry subjects have their origin. The course focuses strongly on distance and angle related subjects; proportion, similarity and area share a relatively low exposure in the text of this part.

This is clear from the very beginning of part I, (Distances, Edges & Regions) where a famous – rather modern – division of a plane area is introduced. The division is natural in situations where there a finite number of points in the area and comparing distances to these points is important in an application. The main idea is the so called Voronoi diagram. Voronoi diagrams are used in many sciences today, from archeology to astronomy and medicine. Basic geometrical

Geometry between application and proof, a general introduction

ideas like perpendicular bisectors, distances, circles spring up here almost by themselves.

Other distance-related subjects can be found in part I too, for instance the so-called iso-distance lines around regions. An example is the famous 200 mile fishery exclusion zone around Iceland. Distance optimisation of routes in diverse situations, with some attention to the Fermat principle, concludes part I.

In part I as a whole one may see a gradual road from application-oriented problems to more pure mathematical thinking. But in this part, the choice of problems is not yet guided by systematic mathematical deduction.

That changes in part II, by far the most ‘pure’ part of the course. Several ideas which originated in the distance geometry of part I, are taken up again in a systematic way. Circles and angles (midpoint and peripheral) play a part in determining Voronoi diagrams, as do special lines like perpendicular and angular bisectors. In part II they will be placed in their proper mathematical environment, an environment ruled by clarifying descriptions and organized argumentation, where one is supposed to use only certain statements in the process of argumentation. This typically mathematical way of handling figures and their relations has its own form of expression: the style of definitions and proofs.

A proof should not be a virtuoso performance by a gifted teacher or student on the blackboard in front of the silent class. It should ideally be found and formulated by the students themselves. This is a heavy demand, which is why, in part II, we pay a lot of attention to the problem of finding and writing down a proof. This also requires some reflection by the students on their own thinking behavior. In the current Dutch situation, 16 and 17 year old students are involved; with them, such an approach can be realized a lot easier than with the students who traditionally read Euclid already at 12 or 13 in the not so remote past.

Important in part II is the stimulating role of Dynamic Geometry Software. We will spend a separate paragraph on it.

Part III (Conflict lines and Reflections) is also connected with part I. A conflict line of two separate regions $A$ and $B$ is the line consisting of the points which have equal distances to the regions $A$ and $B$. It is in a way a generalization of the Voronoi diagram concept of part I and of the perpendicular bisector of two points. The division of the North Sea between Norway, Germany, Holland and Great Britain is a starting example; an important one, because of the oil deposits in the bottom of the North Sea. Later we specialize for simple regions like points, lines and circles. The conflict lines turn out to be good old ellipse, parabola and hyperbola, as characterized by Apollonius of Perga around 200 B.C. Their

---

2. Or conflict set. In the course all of them have the character of lines, so there was not much reason to use the term ‘set’.
properties are studied with distance-based arguments and again with DGS: tangents, directrices, foci and reflection properties.
One of the deep wonders of mathematics is that, as soon as you have clarified your concepts, say from conflict lines in the North Sea to mathematical ones like ellipses and parabolas, those newly constructed ideal objects start to generate new applications by themselves. In part III several old and new acoustic and optical applications of conics are taken in.
In the current 2014 edition of this book an extra chapter focuses on examples of analytic geometry related to distance geometry. This part is specially aimed at connecting analytic and synthetic methods in this field. It is not an independent analytic geometry course. The chapter was not included in the Dutch curriculum in the 1990s, as analytic geometry was put ‘on the back burner’ for about twenty years.

A short note on axioms and deduction

Euclid started *The Elements* with:

*Definition 1: A point is that which has no parts.*

We do not. We start in the middle, where the problems are. Therefore the practice of the Voronoi diagram is used to start building arguments in two directions; downward, looking for basic facts to support the properties of found figures, and upward, by constructing new structures with them. The two directions are called, in the words of Pappos, *analysis* and *synthesis*. Euclid on the other hand, and many of his educationally oriented followers, presented mathematics as building upward only. This course, especially in chapter 2 of part I, indicates that we see axioms also as objects to discover or to construct, not as dictated by the old and unknown bearded Gods of mathematics. A anecdote from the classroom may clarify this:

*At a certain moment, triangle inequality was introduced as a basic underpinning of the distance concept. It expresses the shortest-route idea very well. “Let us agree about the triangle inequality, we will use that as a sure base for our arguments!” But no, this was not accepted by everybody in the class. A group of three students asked: “Why the triangle inequality and not something else?” I countered: “Well, it is just a proposal. By the way, if I propose something else, you will again probably have objections too, don’t you think?” They agreed with that. So I asked: “What would be your choice?” After a few moments they decided on ‘Pythagoras’ as a basic tool to argue about distances. I said, “That’s okay, but there is a problem here: can you base the triangle inequality on ‘Pythagoras’?”*
Ten minutes later they called me again. Yes, they could, and they showed a proof!

The main point of this little story is not the debate over what is an axiom and what is not. It is that these students were actively involved in building up the (or ‘a’?) system itself. It took the mathematical community over two thousand years (from before Euclid to after Hilbert) to build a safe underpinning for geometry in a fully axiomatic-deductive way. It is an illusion to think we can teach students such a system in a few lessons. But we can make them help to become part of the thinking process. Probably we will get no further than local organization of some theorems and results by this approach in secondary education. But we did reach a cornerstone of mathematics in the anecdote above: building by arguments, actively done by yourself.

Dynamic geometry software

Part I of the course includes a computer practical; a program which can draw Voronoi diagrams for a given set of points is used. Such a program allows us to explore properties of this special subject in a handy way. The program that was used in the experiments in the nineties is difficult to run on modern computers running Windows Vista, 7 and 8. It had the advantage that prepared point configurations with some hundreds of points could be read in from a file. On the internet several modern and (a little bit) easier to handle applets are available; maybe you will not find one which can read point sets from file but in the text of this edition we supply drawings of some larger examples to work with on paper. In part II, Dynamic Geometry Software is used with a different purpose in mind. One of the initial exercises was the following. On the screen, draw a circle and a triangle $ABC$ with its vertices on the circle. Construct the perpendiculars from $A$ and $B$ to $BC$ and $AC$ and call the intersection $H$. Now move $C$ over the circle. The lines $BC$ and $AC$, the perpendicular and $H$ will join the movement. Let $H$ leave a trace on the screen during the movement. Big surprise: the trace looks like a circle with the same size as the original one. The student sitting beside me in the computer lab, after lazily performing this construction act: “I suppose we have to prove this?” “Yes, yes, that’s exactly the point. You have to prove your own conjectures, which you found yourself while working with the program!” After finding conjectures on the screen, the question arises again: how to find proofs. A DGS-program does not give any direct hints, it only supplies a adaptable drawing

---

3. For this 2014 edition we adapted tasks requiring DGS from the original *Cabri Géomètre II* to *Geogebra*. 
and offers measuring tools. But especially the changeability and animation of the figures supply all kinds of clues to the careful observer. In the above case you may see for instance that point C is always above H (if you made AB horizontal) and that CH looks constant. So if you can prove that, you are not very far from your goal. Looking for constant elements in an animated construction turned out to be a very good heuristic in finding proofs.

**The aftermath of the Profi project**

The Profi project was performed in close collaboration by teachers and a group of designers at the Freudenthal Institute and overseen by a committee of university researchers. Experimental textbooks were designed, tried out in class and improved. In a later phase, elaborate textbooks were produced by commercial editors, which is the usual approach in the Netherlands. Many aspects of the experimental textbooks illustrated the underlying ideas (which are in a way an upper secondary education elaboration of the theory of Realistic Mathematics Education) much more clearly than the commercial books do, but on the other hand – it should be said also – the commercial books are often better geared to the daily organizational problems faced by the common teacher and student.

Other activities in the Profi project were the design of the so called ‘project tasks’. These are intended for individual or team-use by students, to help them do some independent mathematical research, related to a real life or purely mathematical problem situation. In many cases, students themselves chose DGS as a tool in those tasks.

Each year the Mathematics B-day is organized in the Netherlands, for this group of students. It is a nationwide (recently international) team competition on one day. Students attack (in teams of four) such a problem situation, send in their results, hoping for the honour to be one of the best teams and getting a small prize. The enthusiasm is overwhelming and the number of teams involved is still growing. Students themselves commented that working in-depth on one problem for a longer period is very stimulating for them. Their sound view is not supported by current antididactical trends in education, where subjects are often split up in small digestible bits and mathematics as the activity of building structures disappears almost totally out of sight.

Shortly after the Profi project ended, a major organizational change was introduced in upper secondary education in the Netherlands, the much debated ‘New Second Phase’. Students were supposed to become overnight independent learners, teachers should lay down their supposed superior role of educator in front of the classroom and become counsellors; regular classroom situations were diminished heavily in time.
The contribution ‘Circle and Butterfly’ in this introductory part of the book is an informal report about a regular (but small) class, wrestling with the notion of proof. Our viewpoint there is very clear: learning to prove goes very well by communicating arguments in a debate in the traditional classroom, based on provoking problems; independent student work may be part of the process. The teacher is also a sparring partner in the debate and a guide to help students get some order in their arguments.

The authors

Wolfgang Reuter was one of the teachers involved in the Profi project. At important moments, he put the other designers of the course with their feet on the ground where the students are. His contribution is visible in the careful working out of some task sequences in part I and III. A year after the project ended, Wolfgang died suddenly. Almost all his students came to the funeral. Some of them told moving stories about the way he worked with them.

Martin Kindt did not only design parts of the geometry course for the project, he also gave shape to the calculus course of the project, which breathes the same air. Part of this course is available in a Swiss edition.4

Aad Goddijn has been involved in many curriculum development projects in mathematical education in both lower and upper secondary education.

4. ‘Differenzieren - Do it Yourself’ (ISBN 3-280-04020-5; Orell Füssli Verlag, Zurich). The translated title is in line with what is said above, but beware: the book is in German.
Geometry, classical topics and new applications

Martin Kindt


Geometry is a Greek invention, without which modern science would be impossible. (Bertrand Russell)

Modeling, abstracting, reasoning

The meaning of the word ‘classical’ depends on the context. The classic interpretation of ‘classical geometry’ is ‘Greek geometry’, as described by Euclid. In his work History of Western Philosophy, Bertrand Russell expressed his admiration for the phenomenal achievements of the scientific culture in Greece. Maybe the above quotation gives a sufficient reason why some part of Euclidean geometry should be taught today and in the future and in the future of the future...

To be honest, I must say that Russell is also critical about the Greek approach; he considers it to be one-sided. The Greeks were principally interested in logical deduction and they hardly had an eye for empirical induction.

Lately I found a lovely booklet, Excursions in Geometry by C. Stanley Ogilvy. The first sentence of the first page says: What is Geometry? One young lady, when asked this question, answered without hesitation, “Oh, that is the subject in which we proved things”. When pressed to give an example of one of the ‘things’ proved, she was unable to do so. Why it was a good idea to prove things also eluded her. The book was written in 1969. If it were written in 1996 a young lady in my country, confronted with Stanley Ogilvy’s question, would perhaps answer: “Oh, that’s the subject in which my daddy told me that he had to prove things.”

Stanley Ogilvy very rightly observes that the traditional method of geometry education failed. The things to prove were too obvious to inspire students, the system was too formal, too cold, too bare. In the late sixties, when he published his geometrical essays, the Dutch curriculum more or less skipped the Euclidean approach. Alas, the alternatives such as ‘transformation geometry’ and ‘vector
geometry’ did not fulfil the high expectations. Proofs disappeared gradually, the system (if there was one) was not clear for the students.

Back to the classics in a wider sense. The movies of Buster Keaton may undoubtedly be considered as classics. I remember the famous scene in which he is standing, with his back turned, in front of a house just when the front is falling over, see fig. 1, or better: YouTube. It was a miracle; Buster was standing in the right place, where the open window of the roof landed. The brave actor didn’t use a stand-in. Could it be because Buster had an absolute confidence in geometry? Indeed, with geometry you can exactly determine the safe position!

Make a side view as in fig. 2. The segment $AB$ represents the rectangle on the ground where the falling window will end up.

But is the whole of the rectangle a safe area? Of course not! A man has three dimensions and you have also to take height into account. Fig. 3 shows the side view of the safe area. The Buster Keaton problem gives a good exercise in geometrical modeling for young students:
they have to translate ‘fall over’ as ‘rotate’;
– they have to interpret a side view;
– they have to be aware of the three dimensions of the person (especially of his height);
– they have to combine things, to reason, but... there is no need for formal proof. An interesting question to follow this is: ‘Could the scene be made with a giant?’

As a second example of geometrical modeling I will take the story of a fishery conflict between England and Iceland (in the 1970s). England had a big problem with the extension of the Icelandic fishery zone from a width of 50 miles to a width of 200 miles. In the newspaper we found the picture in fig. 4.

The picture is not only provoking in a political sense, but also geometrically! For instance one can wonder:
– how to measure distances to an island from a position at sea (or vice versa)?
– how to draw the so called iso-distance-curves?
– why is the shape of the boundary of both zones rather smooth compared with the fractal-like coast of Iceland?
– moreover; why is the 200 miles curve more smooth than the 50 miles one?
Introduction

These are typical geometrical questions to investigate. I restrict myself now to the first two questions.

*How to determine on a map the distance between an island and an exterior point?*

You should give this as a, preferably open, question to students (of age 15 for instance).

At some point they will feel the need for a definition. Let them formulate their own definition! After a discussion the class will reach an agreement. For instance: *the distance from an exterior point to the island is the length of the shortest route from that point to the coast.*

This definition is descriptive, not constructive. It does not say how to find the distance, how to determine the shortest route. A primitive way is to measure some routes departing from a given point $P$. In most of the cases you can quickly make a rather good estimation of the nearest point, without measuring all the distances (if... you don’t have too bad an eye for measurements).

More sophisticated is the method using circles. The ‘wave front’ around $P$ touches the island once; the smallest circle around $P$ which has at least one common point with the island determines the distance, as shown in fig. 5.

![fig. 5. Smallest touching circle.](image)

From this idea, the step to the strategy of drawing an iso-distance curve by means of a rolling circle is not a big one. See fig. 6.

Remark: there is an interesting alternative approach of drawing the iso-distance line. The curve arises also as the envelope of the circles with a fixed radius and their centers on the boundary of the island.
Did the Greek geometers have no eye for the aspect of geometrical modeling? They certainly did; Euclid for instance wrote a book about optics (‘vision geometry’). But they differentiated between pure mathematics (the geometry of the philosopher) and practical mathematics (the geometry of the architect). There is an interesting dialogue in Plato between Socrates and Protarchos about the two types of mathematics.

One of the characteristics of the philosophy of Hans Freudenthal is a complete integration between mathematics of real life and so-called pure mathematics. Mathematizing is an activity within mathematics.

In the Iceland case, the fishery conflict can be a good starting point to develop a theory about iso-distance curves of simple geometrical shapes like a quadrangle, to study the difference between convex and not-convex shapes and to make local deductions. There are also opportunities to link this subject with calculus. For instance, it is easy to understand geometrically that in the case of an island with a ‘differentiable boundary’, the shortest route from an exterior point to the island has to be perpendicular to the boundary; see fig. 7.

---

**fig. 6. Circle rolling around Iceland.**

**fig. 7. Shortest route is perpendicular to boundary.**
Indeed, the circle which determines the distance has a common tangent line with the boundary of the island $L$ and the tangent of the circle is perpendicular to the line segment $PP_f$.

In our standards for the math curriculum on pre-university level, the following three important aspects are mentioned:\footnote{Freudenthal respectively used the terms ‘horizontal mathematization, ‘vertical mathematization’ and ‘local deduction’.}

- **Modeling**: the student will get insight in the coherence between a mathematical model and its realistic source;
- **Abstracting**: the student will learn to see that a mathematical model may lead to an autonomous mathematical theory in which the realistic source disappears to the background;
- **Reasoning**: the students will learn to reason logically from given premises and in certain situations, will learn to give a mathematical proof.

**The Dutch geometry curriculum**

The Buster Keaton problem fits very well in the curriculum for the age group 12-15 (‘geometry for all’), it is an example of ‘localization’, which is one of the four strands:

1. **Geometry of vision** (about vision lines and vision angles, shadows and projections, side views and perspective drawings);
2. **Shapes** (two- and three-dimensional);
3. **Localization** (different types of coordinates, elementary loci);
4. **Calculations in geometry** (proportions, distances, areas, volumes, theorem of Pythagoras).

As characteristics of this ‘realistic geometry’ I will mention:

- An intuitive and informal approach;
- A strong relationship with reality;
- No distinction between plane and solid geometry, everything is directed at ‘grasping space’.

The Iceland problem can be extended to a rich field of geometry which I will call here ‘geometry of territories’.

It fits very well within the new geometry curriculum envisaged in the nature and technology profile of pre-university level (age 16–18).

From 1998 we distinguish four profiles in the Dutch curriculum:

- **Culture and Society**;
- **Economy and Society**;
Geometry, classical topics and new applications

– Nature and Health;
– Nature and Technology.

In each of the four profiles mathematics is a compulsory subject, but only in the fourth profile is geometry a substantial part of the curriculum (along with probability and calculus). Over the last three years, we developed (and experimented with) a new program for the ‘Nature profiles’, with attention to:
– the relationship between mathematics and the subjects of the profile (physics, chemistry, biology);
– the mathematical language (how specific should it be?);
– the role of history (mathematics was and is a human activity);
– the use of technology (graphic calculator, software such as Derive or Geogebra);
– the ideas of horizontal and vertical mathematization, local deduction.

I will focus on the geometry part here. We chose the following three strands:
1. Classical metric plane geometry (especially: loci based on distance and angle);
2. Conic sections (synthetic approach);
3. Analytic geometry (elementary equations of loci).

The most important contextual sources in the new program are:
– Territories (conflict lines and iso distance curves);
– Mirrors (focus, normal, tangent);
– Optimization (shortest path, minimal angle).

Involving as main activities:
– Exploring (using computers);
– Modeling;
– Proving (local deduction).

**Geometry of Territories**

The North Sea is divided in national territories. A point in the sea which is equidistant from England (GB) and the Netherlands (NL) is called a **conflict point** of both countries. All possible conflict points form a **conflict line** (or conflict curve or set).

The boundaries of the national territories at sea are parts of conflict lines. Studying a map, it is noticeable that there are ‘three nation points’. For instance, there is one point which is equidistant from GB, NL and DK (Denmark). Students can reason why: the intersection of the conflict lines (GB, NL) and (NL, DK) is a point which is on the one hand equidistant from GB and NL, and on the other equidistant from
NL and DK. Conclusion: the point is equidistant from GB and DK (following the first common notion of Euclid: *things which are equal to the same thing are also equal to one another*). So the intersection point has to be a point of the conflict line of GB and DK. This is a well-known scheme of reasoning; Polya speaks about *the pattern of two loci*.

To simplify things I will study the territories of five small islands (say points) in the ocean, as in fig. 8.

Where the ‘wave fronts’ around two islands meet each other, we have a conflict line. In this case the conflict lines are *perpendicular bisectors*. The fourth picture, without the circles is called a Voronoi diagram. The territories are called Voronoi cells. A boundary between two adjacent cells is called an *edge*. Three edges can meet in one point (‘3 nations point’), such a point is called a *vertex*. The ‘islands’ are the *centers* of the diagram.

Voronoi diagrams (also called ‘Thiessen polygons’) are applied in a lot of disciplines: archeology, geography, informatics, robotics, etc. There is software that makes complicated Voronoi diagrams on the computer screen and students can do a lot of explorations. After some lessons they have a rather good idea of this concept, and then we ask them typical ‘sophistic’ questions (fig. 9).
Why can we be sure that the boundaries are straight lines?
Why do three boundaries sometimes meet in one point?

Analyzing these questions we find two important reasons:
1. Every point of the perpendicular bisector has equal distances to the two points;
2. Every point not on the perpendicular bisector has unequal distances to the two points. The direction of the inequality depends on the side of the boundary.

How to prove 1 and 2?

In the past we used the congruence of triangles (the case SAS). In the age of transformation geometry we used the basic principle of reflection in a line. Our students, who have grown up with realistic geometry, proposed using the theorem of Pythagoras. It is worthwhile discussing these things. You can still ask ‘why’, but at a certain point you have to choose starting points; we call them ‘basic rules’.

A powerful basic rule is the ‘triangle inequality’ from which it follows that a point on the same side as $A$ of the perpendicular bisector of $A$ and $B$, belongs to the territory of $A$, see fig. 10.

With the students we compare two directions of thinking:
In the traditional approach of geometry education we only followed the logical path. That was one of the big didactical mistakes. It is important to show the students (of all levels) the genesis of a piece of mathematics from time to time: the path of exploration (fig. 11). Often the history of mathematics is an excellent
source of inspiration, but in this case I took a rather modern subject, which is a really rich one. The Voronoi theme gives rise to a lot of problems to investigate.

A few examples:
1. Given four points. Make a classification of all types of Voronoi diagrams.
2. Given four points. One point moves along an arbitrary straight line. How does the Voronoi diagram change? (see fig. 12)

3. Given three Voronoi edges, meeting in one point. Can you reconstruct the centers? How many possible solutions are there?
4. Four-nations points are very rare. Can you find a criterion for a such a point? Is

2. We looked for four-nations points in an atlas; only on the map of the USA did we find one: the common vertex of Utah, Colorado, Arizona and New Mexico.
it possible to formulate this in terms of angles?
This leads to the concept of cyclic quadrangle and the theorem of opposite angles.
5. Study the Voronoi diagrams of regular patterns. For instance: twelve points
regularly lying on a circle give a star of rays. If you add a new center (the center
of the circle, fig. 13) a regular polygon arises. Why?
How does the shape of the polygon change if the center moves to the ‘north’?

**The discrete parabola**

Do the same as in example 5 above with a row of equidistant points (fig. 14) the
Voronoi diagram consists of parallel strips. If you add one new center outside the
row, we get an interesting figure:

You could call a part of this pattern a ‘discrete parabola. If we interpolate the row
with more and more points the polygon will tend to a parabola!
In this case we get the continuous version of the Voronoi diagram if we study the
conflict line between a straight line (‘coast’) and a point (‘very small island off the
coast’). Take an arbitrary point on the coast line and draw the perpendicular
bisector between the island and this point. With Geogebra you can move the point
along the coast and see how the perpendicular bisector envelopes a curve (use the Trace-option on the object). The conflict line of island and coast is by definition a parabola and now the students spontaneously discover the property of the tangent of a parabola (see fig. 15), which leads to important technical applications (parabolic mirror, telescope).

In fig. 16 a point $Q$ outside $P$ is drawn on the perpendicular bisector of $L$ and the foot of $P$ ($P_f$).

If a ship is at the position $Q$ it is clear that it is nearer to the coast than to $L$ (for $Q$ has equal distances $a$ to $L$ and $P_f$, and because $Q$ is outside of $P$ this distance is longer than the shortest route to the coast ($= b$)

So every point from the perpendicular bisector, except $P$, belongs to the territory of $C$ and this means that you can say that the line is a tangent of the parabola. Analogously we find a hyperbola or an ellipse in the case where the coast line is a circle, see fig. 17.

You can deduce that $d(P, M) - d(P, L)$ the left picture and $d(P, M) + d(P, L)$ in the right picture are constant (namely always equal to the radius of the circle) and now you know that the conflict line is respectively (a portion) of a hyperbola and an ellipse.

For the students we used the concept of conflict line to introduce both types of
conic section and we confronted them afterwards with the classical definitions. In both cases the perpendicular bisector of \( L \) and \( P_f \) is the tangent.

If the point \( L \) is substituted by a circle (with a radius smaller than the radius of \( C \)), we get the same results.

Now there is a ‘world’ of nice exercises about conflict lines and parabolic, hyperbolic or elliptic mirrors.

\textbf{Some conclusions}

After three years of experiments with students of age 16–18, we are very hopeful that the new geometry curriculum can be realized in a motivating way.

Our experience taught that:

– the students felt challenged by classical geometrical problems, ... provided that these are either introduced by meaningful contexts or are discovered by empirical activities;

– sometimes the students are more critical with proofs than the teacher;

– the use of dynamic geometry software is really a success; students enjoy the dynamic character and they don’t have difficulties with managing the program;

– students are aware of the uncertainty of a discovery by means of the computer; they experience a need to prove non-trivial results.

The geometry stuff is a lucky mixture of ‘old fashioned’ geometry about circles and conic sections and new applications (Voronoi diagrams). Using new technology makes things much more accessible.

While the students for whom this stuff is meant, are much more mature than the students who were confronted with classical Euclidean geometry in the past, making geometrical proofs is attainable. On the other hand, these older students
are less disciplined than the younger ones from the past and this may be a problem when presenting proofs. This last point seems to be the most difficult one. But remember the words of Stanly Ogilvy: to avoid the catastrophe of an uninspired and uninspiring geometry course we will beg the forgiveness of the mathematicians, skip the formalities and take our chances with the rest.

**Literature**


Given: circle with butterfly
or: how do you learn proving?

Aad Goddijn

What came before?

The advanced geometry of mathematics B2-VWO explicitly contains the subject ‘proving in plane geometry’. This article wants to give an impression of how this could work in a classroom. The class involved this article is a small 6 VWO-B group from the Gregorius College in Utrecht (1999); the school belongs to the group of ten schools which have been working with the experimental material of the profi-team. Advanced geometry starts with the book *Distances, edges and domains* (see bibliography). This book proceeds gradually from several applications of the concept of distance – among them Voronoi diagrams, iso-distance lines and optimization problems – towards making proving more explicit. The next part, ‘Thinking in circles and lines’, explains explicitly what a proof is, what you can use in one and how to write one down. The new geometrical material in this part is really geared to the previous book; the theorem of the constancy of the inscribed angle on a fixed arc and the theorem of the cyclic quadrilateral are important. Since these will play an important role in the examples used later on, let us present them in an illustration, see fig. 1.

![fig. 1. Inscribed angles on arc AB of cyclic quadrilateral ABCD](image)

The constant angle theorem says that if $A$, $B$, $C$ and $D$ lie on one circle and $C$ and $D$ lie on the same side of line $AB$, then $\angle ACB$ and $\angle ADB$ are equal.
The twin of this theorem is the theorem of the cyclic quadrilateral. This says: if $A$, $B$, $C$ and $D$ lie in one circle and $C$ and $D$ lie on different sides of $AB$, then the angles at $C$ and $D$ are $180^\circ$ together.

With such building blocks a lot can be done in numerous proofs. Next is an example of what the learning of proving could look like in this stage.

Karin, one of the students in this class, shows in her notebook, fig. 3, that there is something special about the wings of the butterfly shown in fig. 2.

In the proof, ‘angles on the same arc’, that is the named constant angle theorem, has been referred to twice. The idea behind the proof is good, but the execution is not yet perfect: this is called similarity rather than congruence and two pairs of equal angles is enough.

In the course of the learning process solutions and usage of terminology become more accurate and better written.

This needs to be worked on in class, but this is not the key point when it comes to learning to prove. The real problem for Karin and her classmates Sigrid, Janneke, Bas, Mark, Monica, Marleen and Petra is: how do you find a proof in a still unfamiliar situation? And for their teacher Marcel Voorhoeve: how do I help them finding proofs by themselves? The second half of ‘Thinking in circles and lines’ deals especially with this search – and learning how to search – for proofs.

**Form as tool**

A beautiful proof is like a good sonnet: form and content support each other. Example exercise 1 explicitly asks for a proof in a certain format, which has been seen before. Part c refers back to the proof of the concurrency of the three perpendicular bisectors of a triangle $ABC$. Briefly, the proof goes like this. Let the perpendicular bisectors of $AB$ and $BC$ intersect in $M$. Then $d(A,M) = d(B,M)$ holds and also $d(B,M) = d(C,M)$. Connect the equalities and you have that $d(A,M) =$
Given: circle with butterfly or: how do you learn proving?

\[d(C, M)\]. From there, it also follows that \(M\) lies on the perpendicular bisector of \(AC\). The characterization by equal distances of the perpendicular bisector is used, first twice from middle-and-perpendicular to equal distances and then after the connection step once from equal distances to middle-and-perpendicular. The students know this as the 1-1bis form. This form has been assimilated in a scheme (fig. 4) in ‘Distances, edges and domains’.

\[
\begin{array}{c|c}
\text{step 1} & \text{step 1bis} \\
\hline
\text{M on pbs(A, B)} & \text{M on pbs(B, C)} \\
\downarrow & \downarrow \\
d(M, A) = d(M, B) & d(M, B) = d(M, C) \\
\hline
\text{2: connection step} & \\
\downarrow & \\
d(M, A) = d(M, C) & \\
\hline
\text{step 3} & \\
\downarrow & \\
d(M, A) = d(M, C) & \\
\downarrow & \\
M \text{ on pbs(A, C)} & \\
\end{array}
\]

fig. 4.

In example problem 1 (fig. 5) a lot of help is offered; the reference is to the diagram in our fig. 4 and it is even made clear that you should not suppose that the circle through \(B\) and \(C\) goes through the intersection \(S\) of the other two circles. This explication will bear fruit; later on, in a completely different, but very difficult proof one of the students used the phrase ‘You cannot assume that...’

But first Sigrid’s solution (fig. 6). It is clearly structured after the model in fig. 4. The connection step is not explained, it is hardly necessary and it did not fit into the outline: the rest has been explained right above the fragment shown here, including the reverse of the cyclic quadrilateral theorem, which is used in the conclusion.

Writing down the proof in this form was obligatory. However, the student did get the chance to bring in the right ingredients, but did not have to find out in what order they should be mixed. In this phase it is actually not such a bad idea; besides, they learn a special type of proof, which is worth a high place in the repertoire of possibilities.
example exercise 1
In the figure below three equilateral triangles have been put against the sides of triangle $ABC$. The circumcircle of the equilateral triangles seem to pass through one point. This needs to be proven.
Hint: Look back at page 30 and 31 of part A.

This means: find a characterization for points on the small arcs. Call the intersection of those two arcs $S$; show that $S$ lies on the third arc.

a. What is your characterization?
b. Which theorems do you use?
c. Write down the proof in the form of page 30 in part A.
Heuristics

Such a format can be trained and practiced, but for me rules like *if you need to prove that three lines or circles go through one point, then use the outline of 1-1bis* are absolutely not done. This leads to mock results. In this fashion laws are imposed where the student needs to learn to make choices and come up with his or her own plans. Moreover, such rules lead as often to nothing as to real solutions. If one wants to help students finding (or choosing) the form of proofs, the support that is offered must have a more open character. It needs to improve oriented searching, but can never give a guaranteed solution strategy. Such guide rules are also called heuristics. In Anne van Streun’s dissertation *Heuristic math education* he mentions the two just named properties. Van Streun offers a good overview of mathematicians and didactics, both having touched on this subject, and compares several approaches in this area; the mathematically-relevant target area is specified no further than ‘the subject matter of 4VWO’.

Still to be recommended, especially since there are many geometric examples in there for this audience, is ‘How to solve it’ by George Polya. For Polya a heuristic reasoning is meant to find a solution; but the heuristic reasoning is certainly not meant to be the proof itself. Polya’s harder founded ‘Mathematical Discovery’ contains a first chapter named ‘The Pattern of Two Loci’. Some of the heuristics used in ‘Thinking in circles and lines’ can be found there.

In the remainder I will assume the view that some heuristics are very general like ‘make sure you understand the problem, then come up with a plan’ and others are more subject-specific, like the example of the three circles from above. I also would like to show more examples than to preach general theories. Due to the restricted size of the Nieuwe Wiskrant [see http://www.fi.uu.nl/wiskrant, the magazine where this article was published originally] not all named heuristics in ‘Thinking in circles and lines’ will be discussed here. I will not limit my comments on the work of students and teachers solely to heuristics. In an active process of learning a lot of things occur at the same time.

Recognizing

Nearly no one had a problem with example exercise 2, fig. 7, but it does bring some special things to light.

One of those things that novices in proving in plane geometry need to practice is recognizing several familiar configurations within new complex figures. Herein also lies an opportunity for the teacher in the classroom to revisit what is known, or at least should be known. Marcel, the teacher in our case, gratefully used this opportunity regularly. You can doubt whether the students need to know
Introduction

example exercise 2

In this figure three half circles are shown. The diameters of the small circles make up the diameter of the large one. \( BD \) is the common tangent line of the small half circles.

You have to prove: \( DEBF \) is a rectangle.

Proceed as follows:

a. Look for the main theme. It is present in the figure more than once!

b. Now write the proof down yourself in a clear, but not too detailed form.

‘heuristics’ explicitly, but at least for the teacher it is of importance to keep a couple of heuristics in readiness as keys in a learning conversation.

Bas and Mark work together, they have recognized the theme: a right-angled triangle in a half circle, so the theorem of Thales can be used, fig. 8.

\[ \text{fig. 8.} \]

The angles at \( D, E \) and \( F \) are 90°, thus the fourth angle of quadrilateral \( DEBF \) has to be the same. Good, but the first step of the proof, the perpendicularity of \( \angle ABD \) and \( \angle CBD \) now is unpleasantly useless. Mark observed that \( D \) could also lie somewhere else on the great half circle, then \( DEBF \) would still be a rectangle. Then why was the tangent \( BD \) to the small half circles given at all?

\[ \text{fig. 7.} \]
Given: circle with butterfly or: how do you learn proving?

That was a sharp insight! Here one of the facts was redundant. Normally this is not the case in this kind of geometry and this is a good occasion to point out another general heuristic: check during your work whether you used all that was given!

**Learning to note**

The next exercise, in the figure of exercise 2 (fig. 9), was to show that $EF$ is a tangent line of both small half circles. (By the way, in 1996 this question was part of the second round of the Dutch Mathematics Olympiad.) All components of the proof are shown in the figure Monica has drawn, fig. 9.

She writes down the actual proof pretty briefly: the crosses, balls, squares and other things in angles and on line segments do the actual work. Such symbols (and often a complete rainbow of felt-tip pens) come in very handy in the phase of searching for a proof. But it remains draft work, a neat form of noting must be worked on.

Initially students use several angle notations like $\angle ABD$, $\angle A_1$ and mix many symbols, also in the proofs presented to the public. The first two are, in combination with a sketch, acceptable, but the third (the crosses, balls, squares and so on) is not, since the indicated angles are not uniquely fixed, the symbols only indicate the equality in angles, not which angles they are.

There is a good traditional way to improve the correctness of the writing: let them write down a proof in detail, correct it and provide it with personal comments. It
Introduction

takes time, but it pays off; students often develop their own specific notations, which need comments.

Fig. 10 shows a piece of comment given by Marcel Voorhoeve on a piece of Bas’ work (not shown here). Naturally Bas knew what he meant with his notes and Marcel started from there as well, but it looked as if the direction of the logic went in the opposite direction. The comment points out that the arrow is not being used correctly, and is used to introduce an explanation rather than a conclusion.

Find a link

A different heuristic was introduced in example exercise 3, fig. 11 below.

**example exercise 3**

Here two circles are given and two lines \( l \) and \( m \), which go through the intersections \( A \) and \( B \) of the circles.

![Diagram](image)

To prove: \( PQ \parallel RS \).

**Approach:** consider that parallellity and equal angles often go together and look for a link. The circle and the points \( A \) and/or \( B \) of course play a big role.

Students are asked to present their proofs. Volunteer Mark starts his story for the class on the overhead projector after adding a few numbers with: *I am going to*
prove that $\angle Q_2 = \angle S_2$. As far as I am concerned nothing can go wrong: the very general heuristic of ‘know what it is about’ has been applied. Because of this the story to come has a goal and a direction. This follows from earlier class conversations; it often occurs that the story a student tells in class, is a totally dark path with lots of detours for the rest of the class. It is only a matter of time before someone – students or teacher – asks: what on earth are you talking about? These are enlightening moments, since the student in question is often able to say what it is about in one sentence!

The specific heuristic, which under the hood of ‘approach’ is alluded to, is really a totally different one. There is no theorem you can directly apply to show the equality of angles. Thus one needs some intermediate step, object, angle or something else. Despite the rather directive hint in the text of the assignment, which will quickly lead to sketching the help line $AB$, it will take some time before the proof can be seen clearly from the drafts. After six crossed out lines, Mark’s notes look like fig. 12.

![fig. 12](image)

From $Q$ we first go to intermediate stop $A$, and from there to $S$. In the notebook the right angles get as many attention as the usage of cyclic quadrilaterals, but in the explanation the leading role is for the cyclic quadrilateral. This is based on the fact that you can use the angles at $A$ as a link.

Also, there are numerous possible different variations in approach for the students. In essence they all use the same elements, but that is not seen right away. Someone who used $Z$-angles instead of $F$-angles may think that she found a different proof. In this case the class conversation is of great value; the teacher makes clear what the essential line is and what the necessary details are. Thus it came out that the proofs generally differed only in detail.

The strategy of finding links, of which later on an example in a different frame, has a very positive side effect: threshold reduction. A student who suddenly, after doing a lot of exercises, solving equations and working out brackets, is confronted with the proof question in this example sometimes is likely to sigh: well, I don’t know, no idea how I should do it. ‘Find a link’ therefore also means: see if you are
Introduction

able to write something down, even if you do not know up front whether it leads to the solution. After a while you may have enough pieces and even have one or two which match in order to solve your problem. The kindness of the part of geometry on which we are working is that there is so much opportunity for these learning processes, which by the way do not all evolve consciously. The ‘link’ is as old as proving in geometry itself. Book I of Euclid’s *The Elements* contains, after a list of twenty-three definitions and five postulates, the five ‘general rules’ and the first is:

1. *Things, equal to the same, are equal to each other.*

In Euclid a lot of reasoning steps would need some more argumentation according to today’s mathematical standards, but this blindingly obvious platitude is called upon explicitly in crucial parts of proofs. Looked at from a logical perspective, general rule number one formulates an explicitly allowed reasoning step, but in the practice of proving in *The Elements* it is clearly a structuring tool. I consider it a heuristic.

---

**example exercise 4**

Circles $c_1$, $c_2$, $c_3$, and $c_4$ intersect as shown in $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$.

---

To prove: If $A$, $B$, $C$ and $D$ lie on one circle, then $E$, $F$, $G$ and $H$ also lie on one circle.
While solving a proof exercise, the keystone of the vaulting sometimes falls into your hands without prior notice. This is a nice moment, a flood of bright white light is surging trough your head, chaos alters into a pattern and suddenly all lines, angles and circles are made of sparkling crystal. And the path to the proof is wide open and lying in front of you.

Did such heavenly moments occur in the class? Yes, and not so rarely. To some extent Mark has that feeling for a moment when after six lines of bungling he starts over and the proof rolls out in a tight bow. There the watch suddenly has started ticking when the last sprocket was put in and naturally this happens more. The moment often closes a period of frustrated searching, but I do feel that the ‘good feeling’ for many students is more than just the relief ‘Oh, am I glad this is over’. Next is such a situation, which was audible in class.

Example exercise 4, fig. 13, closes the link-section; right before, it has been said that you need more than one link.

Janneke has been working on it for a while in class. Her notes are in fig. 14.

In the four drawn circles, she has made cyclic quadrilaterals and now she is staring at an anthill of numbered, signified and colored angle relations. She needs to show that in quadrilateral $EFGH$ the two opposite angles together make $180^\circ$. But for Heaven’s sake how?

fig.14.
Suddenly a cry: but those also lie on a circle!

Those are the points $A$, $B$, $C$ and $D$. At this time the proof is a done deal for her, she suddenly knows for certain that you need to start with two angles of cyclic quadrilateral $ABCD$; the already found angle relations lead from outside to inside (links!) to a good sum of angles for the two opposite angles in quadrilateral $EFGH$. The breakthrough moment here is the moment where in one flash is seen that there is an unused fact after which the total plan of the proof arises. A scratch through the mess, we are starting to rewrite and the details almost fill out themselves in the computation. This will provide a second bonus: Yeeesss, it is correct!!!

We saw three clear stages in this event: hard work with possible frustration, breakthrough of the insight and getting the verification conclusive.

‘Effort, vision, verification: aspects of doing mathematics’. This was de title of the inaugural speech of Prof. F. Oort in 1968, in Amsterdam. Beautiful to see it so clearly with VWO students!

In Janneke’s notebook, fig. 14, the frustration phase is very easy to recognize in the upper part. It is clear how Janneke (in the figure in her book of course) has numbered the sub-angles. Also, there is no visible direction in the proof and no usage of the relations between the $A$- and $C$-angles. But underneath the line – the moment of insight – it goes really well, the verification is running like water. The first line is confusing for a moment; it still needs to be proven that $\angle EHG + \angle EFG = 180°$, the goal of the computation has been announced here so to speak. Look, at the end this equality returns. The fourth line (the first that starts with 360) contains the joining of the opposite angles $\angle EHG$ and $\angle EFG$; underneath this the key step has been written down in full:

$$\angle A_1 + \angle A_2 + \angle C_1 + \angle C_2 = 180°.$$

Next the deduction is continued by manoeuvering these four angles in the right positions, after which the result follows. ‘thus $H$’ obviously means ‘thus $EFGH$ is a cyclic quadrilateral’. Yes, yes, you need to write down after that that $E$, $F$, $G$ and $H$ thus lie on a circle, but we won’t fall over a trifle right now. It is something, though, that at some point needs to be learned: that you really need to touch the finish line!

**Translating**

In example exercise 5, fig. 15, a by now familiar figure is shown. That the three circles have a point of intersection may be used, since this has been proven. It is useful to let something like this occur explicitly in the learning path; it shows
something of the structure of the course. Students are often prepared in their schoolish kindness to again prove that the three circles intersect. Beforehand it has been said – a heuristic – that sometimes you need to translate that which is to be proven into something else. The translation can be almost the same as the original. For example: isosceles triangles, this is the same as equiangular triangles. Or: three points lie on a line, then two line segments make an angle of $180^\circ$ with each other. Things which are very close to each other, but give one handle more and some more flexibility.

In fig. 16 a fragment from Karin’s work is shown; the remainder of the proof is showing that the angles at $S$ are indeed $60^\circ$ angles. That is simple.

Transcription: You may not assume that $A, S, D$ are on one line. If they are on one line, then $\angle ASD = 180^\circ$. 

---

**example exercise 5**

A familiar figure. The circumcircles of the three equilateral triangles pass through one point. You know that and you can use it.

---

*To prove: $A, S$ and $D$ lie on one line.*
The explication of what still needs to be done, namely showing that $\angle ASD = 180^\circ$, also helps preventing to walk into the trap of already using that form $ASD$ one line. Later this is emphasized on the blackboard by using two different colors for $AD$ and $DS$, an old-fashioned neat classroom trick for the teacher.

**Conjectures and Cabri (or Geogebra)**

In the final chapter the students have to formulate their own conjectures while they are experimenting with Cabri$^1$. These conjectures will be proven later. A special heuristic belongs to this learning method.

In the computer room I am sitting in front of the screen next to Petra and Mark. A circle has been drawn, a triangle lies on the circle with its vertices, so that the vertices can be dragged, while the circle remains fixed. The orthocentre of the triangle has been drawn. Now $C$ is moving over the circle, and therefore $H$ moves as well. Petra let $H$ make a trace; Cabri has an option to do so. What happens? $H$ also moves over a circle. The effect is spectacular when you see it happening and it immediately raises the question: why a circle? This question is a natural motive for looking for a proof.

An important heuristic in Cabri (or other dynamic geometry programs) is: look at the movements on the screen and try to find something that is moving also, but has something constant to it. If you find something like this, you may have a key, maybe a link, in your hands for the proof. This is a good working approach for Cabri and I mention it during the conversation in front of the screen.

Petra answers that $\angle AHB$ is constant and shows that she also sees the constancy of $\angle C$. The bell interrupts and all I can say is ‘you can do this’. I feared that Petra looked at it a little bit differently: she may have deducted the constancy of $\angle AHB$ from the fact that $H$ lies on a fixed circle through $AB$. That is assuming what you need to prove, the most deadly sin there is in mathematics.

She has worked it out on paper at home, and the result surprised me. Fig. 17, next page, is from Petra’s notebook. I was wrong, or Petra changed her mind. Look at how Petra starts her argumentation: *if the path of $H$ is a circle, then $\angle AHB$ must be constant, so we will prove it to be*. That is the old heuristic, which Pappos named ‘analysis’: exploring the problem from the assumption that we have the solution. Next should be the synthesis-phase: constructing the proof from the given, the opposite direction of the analysis. Petra’s synthesis starts at the fixed angle $C$; a long detour of almost a page in which – how else – we encounter cyclic quadrilateral $CFHE$, leads her via $\angle AHB = 180^\circ - \angle ACB$ to the required result.

---

1. Note (2014): In 1999 *Cabri Géomètre I* was used. Any current Dynamic Geometry Software program will support the options used in the text.
Petra must have enjoyed this success; she closes very professionally with step 13, (fig. 18) where it becomes clear that she acknowledges the (not at any cost necessary) case distinction.

Translation: if C is on the other side, it is the same.
Other side: down under $AB$, yes.

**Finally: teacher and student**

During this story I have pointed out several times that heuristics also belong to the conversation tool kit of the teacher. The heuristics then are an aid to finding coherence in the search process. The idea here is not about one heuristic being better than another, but about the direct effectiveness with respect to the content
of it all. The emphasis lies on stimulating the search so that one will no longer say: I do not know that now, so I cannot do it. This is achieved in this little paradise class.

Teacher Marcel Voorhoeve also takes on other roles besides the organizational-directing one: the role of co-solver and also that of critical sounding board via questions dealing for example with half-grown proof steps and sometimes students take over that last part in conversations. The task of demonstrating on the blackboard is hardly of any importance; as far as learning to find proofs goes, it does not seem very effective, and directing towards acceptable ways of noting down could also be learned through the students’ work.

Sieb Kemme and Wim Groen have written in the Nieuwe Wiskrant (19,2) about problem solving as a trade. After their introduction, I of course started to deal with their example problem in a different manner, but also their reflections on the search process matched my approach and agree with what I have brought up here. Reflecting independently is something I see Sieb, Wim and myself however, naturally – or just because of age and professional knowledge – do more than the students in this 6 VWO class; in 6 VWO such things are more open for discussion than in say a 3 grammar school class. Here again lies a task for the teacher.

**Geometrical footnotes**

Sometimes to keep on solving a problem yourself and to look how you do that, remains an important exercise for those who have to teach those things. So why not add a couple of nice continuations of one of the problems presented in this article?

1. The point $S$ in example exercise 1 is the first point of Fermat, $F_1$, although the Italians will keep it calling the point of Torricelli. Reflect the triangles also to the other side of the sides; show that the three circles then also pass through one point $F_2$. Use the plagiarism-heuristic: detailed copying of a proof with some small changes.

2. In example exercise 5 $AD$, $BE$ and $CF$ all three of course go through the point $S$ (of $F_1$). But those three segments also have the same length. This should not be difficult to prove, especially for those whose memory of a previous phase of the geometry education (transformations) is still vivid.

3. Plagiarize exercise 2 like exercise 1 plagiarizes example exercise 1.

4. In example exercise 5 it was proven that $AD$, $BE$ and $CF$ go through one point, for the case that the outer triangles are equilateral. Now put three isosceles, mutually uniformly, triangles with their bases on the three sides of $ABC$ and now
Given: circle with butterfly or: how do you learn proving?

prove also that $AD, BE$ and $CF$ go through one point. You need to forget about the circles! This exercise may be seen as more difficult.

5. Draw a triangle (use Geogebra for instance). Construct both Fermat points, the center of the circumcircle and the center of the nine points circle and show that these four lie on one circle. J. Lester has showed this remarkable relation in 1995. Heuristic: use someone else’s work from the Internet to find a proof.

**Literature**


Distances, edges and domains

Advanced geometry, part I
Chapter 1: Voronoi diagrams
In this first part of the geometry course you will encounter a way of partitioning an area which has many applications. We will start with simple cases, but make sure you keep an eye on the figure on the front page. That is also an example of one of the partitions in this chapter!

You will have to sketch quite a lot. You can do that in this book. For some exercises special worksheets are included. You will usually need a protractor, ruler and compass. You may also have to make sketches and graphs in your notebook while working out the solutions.
1. **In the desert**

   Below, you see part of a map of a desert. There are five wells in this area. Imagine you and your herd of sheep are standing at \( J \). You are very thirsty and you only brought this map with you.

   1. a. To which well would you go for water?
      b. Point out two other places from where you would also go to well 2. Choose them far apart from each other.
      c. Now sketch a division of the desert in five parts; each part belongs to one well. It is the domain around that particular well. Anywhere in this domain that special well must be the nearest.
      d. What can you do when you are standing exactly on the edge of two different domains?
      e. Do the domains of wells 1 and 5 adjoin? Or: try to find a point which has equal distances to wells 1 and 5 and has larger distances to all the other wells.
      f. In reality the desert is much larger than is shown on this map. If there are no other wells throughout the desert than the five on this map, do the domains of wells 3 and 4 adjoin?
      g. The edge between the domains of wells 2 and 3 crosses the line segment between wells 2 and 3 exactly in the middle. Does something similar apply to the other edges?
      h. What kind of lines are the edges? Straight? Curved?

   In this exercise you just partitioned an area according to the *nearest-neighbor-principle*. Similar partitions are used in several sciences, for instance in geology,
Part I: Distances, edges & domains

forestry, marketing, astronomy, robotics, linguistics, crystallography, meteorology, to name but a few. We will revisit those now and then. Next we will investigate the simple case of two wells, or actually two points, since we might not be dealing with wells in other applications.

2. The edges between two domains

folding
A simple case with two wells is shown here. We neglect the dimensions of the wells themselves, i.e.: we pretend they have no size at all. In the figure: the points A and B. On paper the edge between the domains of A and B is easy to find, namely by folding the paper so that A lies on top of B. This folding line is the edge between the areas belonging to A and B.

protractor
There is also another method to find this edge easily and fast: with the protractor. See the figure on the right. A and B are both at the same distance from the middle of the protractor.

half plane
In this figure the areas round A and B have different colors. To the A-domain applies: distance to A < distance to B. To the B-domain applies: distance to A > distance to B. Only on the edge the following applies: distance to A = distance to B.
Actually, you should imagine that there is more than just the sketched part: everything continues unlimited in all directions. The two domains, determined in such a way, are infinitely large and are bounded by a straight line. The name for such a domain is half-plane. Include the edge as part of both half-planes. In the figure the domains of $A$ and $B$ are therefore both half-planes. These half-planes overlap each other on the edge.

2. The edge is often called conflict line. A good name? Why?

3. More points, more edges

Through folding, we will now investigate a situation with four points. Also take worksheet A: Folding to Voronoi (page 175).

3. For each pair of points we determine the edge by folding.
   a. First find out how many folds are necessary and then proceed with the actual folding. Try to do this as precise as possible; for instance hold the piece of paper up to the light. Use the folded lines to sketch the partition of the area.
   b. While folding, a lot of intersections of the folding lines arise. Nevertheless there are different kinds of intersections. What differences do you notice?
   c. One fold turns out to be redundant. What causes that?

excluding technique

Here, a situation with five points. The edges for all pairs of points are shown.

4. How many edges are there?
   a. Of the cross $x$ near the edge $BD$ you know for certain: it certainly does not belong to $B$. How can you tell?
   b. Use other lines to exclude other possible owners of $x$. In the end one remains. Which one?
   c. Try to find out for the other areas to what center they belong and with this excluding technique finish the partition using five colors.
Part I: Distances, edges & domains

**Exact Voronoi diagram with the protractor**

5. Now sketch, using the protractor method, the exact edges round the wells on the desert map in *worksheet B: Exact Voronoi diagram for the desert*, page 176.

4. **Voronoi diagrams: centers, edges, cells**

In this paragraph we discuss some more terminology.

**nearest-neighbor- principle**

In the preceding we made partitions of an area according to the ‘*nearest-neighbor-principle*’.

**centers**

The points around which everything evolves (in this example the wells) will now be called *centers*. Throughout this book we will always assume we have a finite number of centers.

**Voronoi diagram**

The figure of edges is called the *Voronoi diagram* belonging to the centers. Another name is *edge diagram*.

**Voronoi cell**

The area that belongs to a center, is called a *Voronoi cell*, or, in short, *cell* of that center.

**vertices**

A Voronoi cell is bounded by straight lines or by segments of straight lines. The points where several lines converge, are called the *vertices* of the Voronoi diagram (singular: *vertex*).

**history**

Voronoi diagrams are named after the mathematician Voronoi. He (in 1908) and Dirichlet (in 1850) used these diagrams in a pure mathematical problem, the investigation of positive definite quadratic forms. In 1911, Thiessen used the same sort of diagrams while determining quantities of precipitation in an area, while only measuring at a small number of points. In meteorology, geography and archaeology the term Thiessen-polytope instead of Voronoi cell became established.
6. a. On the next page you can see six situations. Each dot represents a center. Sketch the edge diagrams for these situations.

b. In situation I you find one point in the middle where three edges converge. What can you say about the distances of that point to the centers?

c. Does situation II also have such a point?

d. In the situations III and IV only one center is not in the same place. However, the Voronoi diagrams differ considerably. Try to indicate the cause of that difference.

e. In situation V the centers lie on one line and thus the diagram is fairly easy to draw. What can you say about the mutual position of the edges and the shape of the Voronoi cells?

f. Situation VI has lots of centers. But thanks to the regularity, sketching of the edge diagram is again a simple affair. Once one cell is known, the rest follows automatically.

Do you know anything in nature which has this pattern as a partition?

Infinitely large cells

7. Nowhere is it said that a Voronoi cell is enclosed on all sides by (segments of) lines. In fact, some cells are infinitely large, even though that is not visible in the picture.

a. How many infinite cells are there in the well example on page 43?

b. In situations with two or three wells there are only infinite cells. Now sketch two situations with four centers. In one situation all cells must be infinitely large, in the other situation not all cells are infinitely large.

c. Describe a situation with twenty centers and twenty infinite cells.

d. Where do you expect the infinite cells to occur in a Voronoi diagram?
5. *Three countries meeting; empty circles*

Below you see a redivision of the Netherlands as a Voronoi diagram\(^1\). The centers are the province capitals.

![Voronoi diagram of the Netherlands](image)

**three-countries-point**

On each of the vertices of the Voronoi diagram, three cells converge. This is what we call a *three-countries-point*, even in a context that does not involve countries.

8. a. What do you know of the distances of the ‘three-countries-point’ between the cities of Middelburg, Den Haag and Den Bosch to those three cities?

b. Put your compass point in that three-countries-point. Now draw a circle through those three cities with this three-countries-point as its center.

c. Now put your compass point somewhere on the edge between Zwolle and Arnhem, but not in a vertex of the diagram. Sketch a circle with this point as center, which passes through Arnhem.

---

1. Use worksheet C: map of the Netherlands, page 177, to see the official division in provinces.
largest empty circles
What you just sketched, are examples of largest empty circles. A largest empty circle in a Voronoi diagram is a circle in which no centers lie and on which lies at least one center.

The name largest empty circle is chosen well: if you enlarge such a circle around its center just a tiny bit, the interior of the circle would not be empty anymore: for sure there will be one or more centers inside.

9 a. In this Voronoi diagram two largest empty circles are already sketched. Mark their centers.

b. Sketch several largest empty circles:
   - with four centers on the circle,
   - with two centers on the circle,
   - with one center on the circle.

c. What can you say in general about the number of centers on a largest empty circle round a three-countries-point?

10 a. On the right you see a situation with four centers. The centers are the black dots. The little star at $M$ is the center of the circle through the centers $A$, $B$ and $C$. Can $M$ represent the three-countries-point of the cells around $A$, $B$ and $C$? Why? Or why not?
b. Here you see the same figure, only the center $D$ is left out. Sketch the Voronoi diagram. Be careful: $M$ is not a center itself, but you can make good use of $M$ in some way.

c. Now add center $D$ yourself and expand the Voronoi diagram, but do it in such a way that $M$ becomes a four-countries-point.

d. Three-countries-points are very normal, four-countries-points are special. Explain why.

11. On the next page you will find a Voronoi diagram of which the centers lie on the coast of four islands. Parts of the Voronoi diagram have actually become edges between domains around those islands.

a. Mark those edges with a color. A partition in four domains arises.

b. You could also talk about three-countries-points in the last partition. Now sketch a couple of largest empty circles, which just touch three of the islands. Where should you place their centers?

true four-countries-point
True four-countries-points occur very rarely between countries in real life. One is shown on the right, between the states Utah, Colorado, Arizona and New Mexico in the USA. If you know another one, speak up!

Of course this is not Voronoi diagram of the United States.
12. Sketch a situation of centers and a *Voronoi diagram* yourself, for which:
- only four-countries-points occur and no three-countries-points
- and for which there is a square cell
- and for which the edges lie in each other’s extension, are parallels, or are perpendicular.
6. Chambered tombs in Drenthe

A partition of the eastern part of the Drents plateau in imaginary territories. Centers here represent groups of chambered tombs\(^2\). Some chambered tombs were used to store bones and skulls over a period of 600 years. The Voronoi diagram shows a possible partitioning of the area. Archeologists often research whether such partitions correspond to the distribution of pottery in an area. This could provide indications about the social and economical structure in former days.

13 a. Observe: the cell in which Assen lies and the one north of it have centers which lie symmetrical in relation to the edge. Why that symmetry?

b. Do the centers lie symmetrical everywhere in relation to the borders? Is this necessarily so for a Voronoi diagram?

c. The cell southwest of Assen contains several dots. Which dot is used for making the Voronoi diagram?

---

2. From about 3500 B.C. They are called ‘hunebed’ in Dutch.
14. Reasoning with symmetry could also help to fill up an incomplete map of centers and edges. On the right an incomplete Voronoi diagram is shown. Complete the diagram.

**reflection**

Voronoi centers of adjacent cells are always each other’s mirror image in relation to their Voronoi edge. So you can recover missing centers by reflecting in an edge! We will illustrate this technique with the following two examples.

15. This figure shows only one center. Since two edges are (partially) indicated, there have to be two other centers and a third edge. Finish the sketch accurately.

16. Below, the edges of a Voronoi diagram with three centers are given. Point $P$ lies in cell $a$.

*Try to work as precise as possible in this exercise, or you may run into trouble. You can do the reflection exactly using your protractor. See I - 44.*

![Diagram](image)

**a. $P$ is certainly not the center which belongs to cell $a$!**

You can verify this by reflecting $P$ in edge I; name the reflection $P_1$. Then reflect $P_1$ in edge II. Name the reflection $P_2$. Finally, reflect $P_2$ in edge III. Name the reflection $Q$.

Why is it not possible that $P$ is the center of cell $a$?
b. Sketch the middle of the line \( PQ \) and name it \( R \). Now also reflect \( R \) successively in the three edges; in this way point \( S \) arises. What do you notice about this ultimate point \( S \)?

c. The point \( R \) (or \( S \)) could be the center of cell \( a \), but that is not necessarily true. Another option, for example, is a point that lies in the middle of \( R \) and the indicated three-countries-point \( M \). Verify this by repeated reflection. What other points could you take to start with as center of cell \( a \)?

**reconstruction problem**

The final result of exercise 16 is surprising. Using the method given above, you are apparently able to reconstruct possible centers, without knowing one of them. Of course the question is: Why does this work so well? Several clues are given in the extra exploration exercise, on page 58, so you can get to the bottom of this.

**Summary of chapter 1**

This chapter provisionally explored Voronoi diagrams, and discussed several concepts.

**nearest-neighbor-principle**

A Voronoi diagram arises when a number of points are given and the plane is partitioned so you can determine everywhere what the nearest point is. In this fashion a partition in subregions arises. This is called partitioning according to the nearest-neighbor-principle.

You will find definitions of the concepts center, Voronoi diagram, edge diagram, Voronoi cell, cell and vertex on page 48.

The Voronoi cells can be infinitely large. These infinite cells belong to centers which lie close to the side; we shall have to specify this later.

**three-countries-points**

In general three cells meet in a vertex. Such vertices are called three-countries-points. To have more than three cells converging in a vertex is possible, but rare.

**largest empty circle**

A circle in which no centers lie, but on which does lie a center, is called a largest empty circle. Such a circle cannot be enlarged from its center.

**reflection, reconstructing problem**

If only edges are given the centers can sometimes be recovered by reflection. For this, the fact that the centers lie symmetrically with respect to their edge is used.
To find the centers for a given Voronoi diagram is called solving the reconstruction problem.

**Preview**

In the next chapters we will first go deeper into the mathematics, which until now we used incidentally. Doing that, we will argue more independently and draw fewer conclusions from measuring figures only. Nevertheless one result will be very practical, namely that we will find a fast way to determine whether a center $D$ is inside the circle through $A$, $B$, $C$ or not, without determining the actual circle itself. This will make the construction of a Voronoi diagram a lot easier. In a later chapter you will learn more about Voronoi diagrams using a computer program. For this you will need the knowledge in this chapter as well as the next.

**Extra exploration exercises: Recovering the centers**

This exercise combines with exercise 16. First of all, make sure that you understand the method used to construct possible centers of a Voronoi diagram of three cells with a three-countries-point.

**Exercise one**

Find and describe an argumentation which proves that the method of exercise 16 always works. Several hints follow here.

a. The figure on the right already shows $P_1$ and $P_2$. The next reflected point would be $Q$, but for now call this point $P_3$ and do three more reflections. You will discover something very special about $P_6$ if you have drawn carefully enough.

b. If you were sure that indeed $P_6$ is always the same as $P$ is always true, then you can conclude that the middle $R$ of $PP_3$ will end up on top of itself after three reflections. Find out why by reflecting the line segment $PP_3$ three times.
c. But why is $P_6$ equal to $P$? That’s the main question now.
The figure shows several angles. You could also think about reflection as the rotation of the bar $MP$ around the center of rotation $M$. Compare the angle over which $MP$ has to rotate (clockwise) to arrive at $MP_2$ with the angle of cell $b$.

**Exercise two**
Once you found one possible position for the center of cell $a$, you also know all the other possibilities. Work this out.

**Exercise three**
Find a method to recover the four centers that are not given in the same type of Voronoi diagrams as shown below. Hint: Divide and rule.

*Complete your investigation as follows:* Make a report of one page in which you write down your argumentations. Add clear figures.
Part I: Distances, edges & domains
Chapter 2: Reasoning with distances
In this chapter the reasoning will be a lot more exact than in the previous chapter. In principle we deduct things from scarcely any given data. We will also consider how that kind of reasoning works and how to write it down.

The illustration with the text in gothic letters on the front page becomes a reality: \textit{UNTERWEYSUNG DER MESSUNG MIT DEM ZIRCKEL UND RICHTSCHEYT}. This is a geometry book, published in 1525 for painters and artists, by Albrecht Dürer. The figure is an illustration for what in this chapter will be \textit{theorem 5}. That theorem says that for any triangle there is exactly one circle which passes through the three vertices of the triangle. In his figure, Dürer uses the triangle $abc$, and indicates how the center of the circle and the circle itself can be determined by ruler and compass only.
7. **Introduction: reasoning in geometry**

1. In the last chapter we investigated several things in relation to Voronoi diagrams. We used what could be seen in figures and sketches. However, we might have assumed things that we do not know for sure whether they are true.

**problem A**

Why does the Voronoi diagram of three points always looks like this?

![Diagram](image1)

and never like this?

![Diagram](image2)

**problem B**

The Voronoi edge of two points always appears to be a straight line. The folding technique backs up this idea. But why are folding lines always straight? We have seen this in many cases. We do not really know why.

In this chapter we will look further into these questions.

**reasoning instead of observing**

Now we want to get certainty about these questions by reasoning in general situations, and not by observing a special case in a sketched figure. This is why this chapter will have a more theoretical character, especially towards the end, since we will start with concrete diagrams, but in the latter part theorems and proofs are discussed.

You may feel as if you are walking on egg shells. This is true, but you will get used to it. Moreover, you will start complaining if something is stated without a proof.

1. **a.** In the section ‘The edges between two domains’ (page 46) more non-found properties of the edge between two areas are stated. Which for example?

   **b.** And how do we use those in the section ‘Chambered tombs in Drenthe’ (page 55)?
8. **An argumentation about three-countries-points and circles**

First we tackle problem A above: *Why do the edges of a Voronoi diagram with three centers intersect in one point?*

You need to look at such questions with a critical eye. Hence:

1. **a.** Sketch a few three-centers situations, which don’t even have three edges.
   - Thumb through the examples of the previous chapter, if need be, to get some ideas.

   **b.** What characterizes these situations?

In the remainder of this section we will not reckon with this special case.

In the previous chapter the Voronoi edge played the leading role. The figure shows its characteristics.

Only the points on the edge have equal distance to \( A \) and \( B \). Phrased differently:

**property Voronoi edge**

\[
\text{(distance from } P \text{ to } A) = \text{(distance from } P \text{ to } B) \\
\text{applies only to the points } P \text{ on the Voronoi edge between } A \text{ and } B
\]

We will use this clear form while reasoning with three edges. Below you see a figure for which it is not known whether the three edges converge into one point.

3. **This exercise will help you find an argumentation for:**

   *the three edges converge in one point.*

   **a.** Indicate the point of intersection of Voronoi edge \( AB \) and Voronoi edge \( BC \) and call it \( M \).
b. Write down, using the property of Voronoi edges, the two accompanying equalities and derive the third equality. Write that one down as well.

c. What does that equality say about point $M$? (Remember the property once more.)

d. Did you reach the goal of the argumentation?

4. The circle, which has $M$ as its center and passes through $A$, also passes through $B$ and $C$.

a. How do you know that for certain?

b. In the previous chapter this circle played an important role. What role was that?

critical remarks

While working on exercises 3 and 4 we solved the problem on page 59, and more, or so it seems.

5. Try to answer the following questions:

a. On what is the assumption that such an intersection point $M$ exists based?

b. Do we not also (maybe carefully hidden, but nevertheless) use the fact that the edges are straight lines?

This is not as easy as it looks!

However, the argumentation of exercise 3 is beautiful, and we will hold on to it. From here on, though, we choose to look for more certainty.

We will follow this strategy:

a. Show unimpeachably that the Voronoi edge of two centers is a straight line.

b. Show that under the condition that the three centers are not on one line, both edges do intersect.

In this fashion we will look for a solid foundation in our argumentation. That has to be found in the properties of the concept of distance, because that is where it all began. In the following paragraph we will work towards this solid foundation, starting with a possibly unexpected problem.

9. **Shortest paths and triangle inequality**

shortest-path-principle

The shortest path from point $A$ to point $B$ is the straight line segment, which connects point $A$ with $B$.

6. But what is the shortest route (figure on next page) from $A$ to $B$ if during the we also have to go via line $l$ like in the figure on the next page? We will get to the bottom of this now.
Part I: Distances, edges & domains

a. Measure which of the three paths from A to B is the shortest.

b. We don’t know whether there might be an even shorter path! Here is a pretty trick:
   Reflect A in line l, name the reflection A’.
   Also connect A’ to the points P.
   Why does the following now apply:
   from A to B via P₁ is as long as from A’ to B via P₁?

c. Now determine, using point A’, point Q on l, such that the path via Q leads to the shortest path.

d. Think of a situation where it is of importance to find a shortest path of this kind.

There is much more to discover about finding shortest paths in more complex situations. We will do so in chapter 6. For now we only establish that the shortest-path-principle is the basis of the solution. We will rephrase this principle more precisely. First, since we will be talking about distance all the time, we introduce a notation for the distance between two points.

distance notation
From here on we will denote the distance between two points A and B as d(A, B). Because we are thinking in terms of comparing distances, it does not matter whether you think of centimeters on paper or of kilometers in the landscape. d(A, B) is always a nonnegative number and you can use it in expressions such as equalities and inequalities. Also expressions like d(A, B) + d(C, D) have meaning. The d originates from the word distance.

simple properties
7  a. Translate in common English what is asserted here:
   The following always holds for points P and Q : d(Q, P) = d(P, Q)
b. What can you say about points \( A \) and \( B \) if \( d(A, B) = 0 \)?

The figure shows three points and their connections:

Next we will describe, using the d-notation, that going from \( A \) to \( C \) via \( B \) is a detour when \( B \) is not on the line segment \( AC \). This has a name: the triangle inequality.

**Triangle inequality**

For each set of three points \( A, B \) and \( C \) holds: \( d(A, C) \leq d(A, B) + d(B, C) \).

The equality occurs only if \( B \) is on the line segment between \( A \) and \( C \).

In every other case there is a true inequality.

The name *triangle inequality* originates from the fact that the inequality holds if \( A, B \) and \( C \) form a triangle.

8. In the figure on the previous page, expanded with \( A' \) and \( Q \), you can apply the triangle inequality to show that \( A-Q-B \) is the shortest path.

a. To which triangle would you apply it?

b. \( Q \) is on \( A'B \); this is drawn. What can you say about the triangle inequality in this situation?

9. On closer inspection the triangle inequality is somewhat more modest than the shortest-path-principle.

a. Sketch a situation in which two paths are compared, and where the shortest-path-principle has some meaning, but the triangle equation has not.

b. The argumentation of exercise 6 runs impeccably and yet only the triangle inequality is used. What causes that?

Of course we choose the simplest principles as starting-points for argumentation.

**starting-point for argumentation**

We therefore assume the triangle inequality as an established fact. From now on, you can refer to it in your argumentations.
10. You could legitimately ask yourself: What is the triangle inequality itself based on? If you asked yourself that question, the following exercise is for you, otherwise not.

a. Good question, difficult answer. Try to think of something yourself on which you would base the triangle inequality.

b. In that case, what would be the next question?

A different notation

Often the notation $|AB|$ is used for the length of a line segment with end points $A$ and $B$. In this chapter, where everything breathes the air of distances, we use the notation $d(A, B)$. When we encounter figures with line segments and their lengths, you will also see $|AB|$.

extra exercise

11. Show that for every set of four points $A$, $B$, $C$, and $D$ holds:

$$|AD| \leq |AB| + |BC| + |CD|.$$

10. The concept of distance, Pythagorean Theorem

Another very important property of the concept of distance can be expressed as the well-known Pythagorean Theorem for right-angled triangles.

12. Phrase that theorem using the $d$-notation of the previous section. Your phrasing should deal with a triangle $ABC$, of which one angle is right.

We will now use this theorem in order to determine the shortest distance from a point to a line and also to ensure the correctness of the method.

shortest distance to a line

In this figure $A$ is a point outside the line $l$. You are probably convinced of the following:
Chapter 2: Reasoning with distances

Of all possible connection line segments the line segment, which is perpendicular to $l$ is the shortest.

13 a. Write down – in $d$-notation – what holds for triangle $APQ$ according to the Pythagorean Theorem.

b. How does $d(A, P) < d(A, Q)$ result from that?

The Pythagorean Theorem is also one of the fundamentals you can use. You could also prove the Pythagorean Theorem based on more elementary things, but we will also not do this exhaustively. One possibility is outlined below as an ‘extra’.

**extra: a proof by Multatuli**

(This is nr. 529 from part II of the Ideas by Multatuli (1820–1887))

I recently found a new proof for the Pythagorean theorem. Here it is. By, as shown in the adjacent figure, constructing six triangles – each equal to the given right-angled triangle – one acquires two equal squares, $AB$ and $CD$. If one subtracts four triangles of each of these figures, one proves the equality of the remainder on both sides, which was to be shown.

It cannot be done any simpler, or so it seems to me. After finding this proof, I heard of the existence of an article, which discusses this topic. I purchased this little book, but it did not contain my demonstration. Furthermore I deem that none of the therein assimilated proofs is as illustrative and clear as mine.

Thus far the proud writer of the *Max Havelaar*. 

69
Part I: Distances, edges & domains

Multatuli’s proof leans heavily on the concept area. We did not exactly establish what its properties are. Moreover, it is rather easily assumed that certain parts of the figure are squares. But very well, for now we will join Multatuli.

14 a. Put some more letters in the figure and write down an argumentation which eventually leads to the equality part of the Pythagorean Theorem, expressed in areas of certain squares.

b. What would you have to show in order to conclude that the oblique ‘square’ is in fact a square.

extra; alternative for exercise 13 a/b

15. You can also prove without the Pythagorean Theorem that the perpendicular line from A to l provides the shortest distance. Use the following hint and your own inventiveness.

Hint: How do you get from A to A the fastest if you have to go via line l?

Pythagoras in a medieval monastery

The illustration below comes from a medieval manuscript. It was made in the monastery of Mont Saint Michel in Bretagne, when Robert de Torigni was the abbot, during the years 1154 through 1186. The manuscript contains figures and texts about astronomy; the abacus, bells, and of course mathematics are used in each of those. Presumably a lot is copied from Arabic manuscripts; in the Arabic world of those days a lot more attention was paid to mathematics and science than in Christian Europe.

This picture is of an application of the Pythagorean theorem in archery. You can see the arch of the bow, and the word ‘sagitta’ (arrow) is written at the hypotenuse and the base. You can also see close to the sides: ‘filum V pedii’, ‘filum IIII pedii’ and ‘altitudo III pedii’. Translated: threads of 5 and 4 foot, a height of 3 foot. It is the well-known 3-4-5 triangle.
11. Properties of the perpendicular bisector

Next we will use the triangle inequality to prove that the Voronoi edge of the centers $A$ and $B$ is equal to the perpendicular bisector of $A$ and $B$.

important

This section is definitely the hardest of this chapter. Even if you do not catch on to all the details, you will be able to continue with the next section, but make a good effort to try to follow the reasoning. The more argumentations like this you can follow, the easier it will get later on, simply because you have had some training.

First a definition, which should be familiar. With definitions in mathematics we establish exactly what is meant.

**definition Voronoi edge**

The Voronoi edge between two points $A$ and $B$ is the set of points $P$ for which hold: $d(P, A) = d(P, B)$.

In chapter I you got the big impression that Voronoi edge of $A$ and $B$ is exactly the perpendicular bisector of the line segment $AB$. We will now prove this. Since we want to start only from definitions and familiar things, we also have to define ‘perpendicular bisector’.

**definition perpendicular bisector**

The perpendicular bisector of line segment $AB$ is the line which is passing through the midpoint of $AB$ and is perpendicular to $AB$.

$pbs(A, B)$

We will agree upon a notation for the ‘perpendicular bisector of line segment $AB$’: $pbs(A, B)$. The figure displays two characteristics of the perpendicular bisector:

- it is perpendicular to the line segment
- it divides the line segment in two.
Part I: Distances, edges & domains

What we would love to pose is the fact that the two concepts of Voronoi edge and perpendicular bisector are actually one and the same. So we state:

**statement of equality**

The Voronoi edge of two points \( A \) and \( B \) and the perpendicular bisector of the line segment \( AB \) coincide.

This means quite a lot: not only that all of the points of the perpendicular bisector lie on the Voronoi edge, but also that the Voronoi edge does not consist of more points. And vice versa. That is why two things need to be proven separately:

a. Every point which lies on the perpendicular bisector, is also on the Voronoi edge.

b. Every point which does *not* lie on the perpendicular bisector, is also *not* on the Voronoi edge.

We will discuss both parts separately.

16. Proof of part a:

Every point, which lies on the perpendicular bisector is also on the Voronoi edge. The figure shows line segment \( AB \) and also the \( \text{pbs}(A, B) \). \( Q \) is the middle of \( AB \). \( P \) is a point that lies on the perpendicular bisector.

a. Indicate in the figure, in green, the two things, which you can use now according to the definition of \( \text{pbs}(A, B) \).

b. Color the line segments of which you have to prove that they have the same length red.

c. Write down what Pythagoras says about \( d(P, A) \) and \( d(P, B) \) and derive from that:

\[ d(P, A) = d(P, B). \]

d. Did you use both of the characteristics of the perpendicular bisector? Where in the argumentation?
17. We are halfway there, but we still have to do the proof of b:

Every point, which does not lie on the perpendicular bisector, is also not on the Voronoi edge.

See the figure above. $Q$ is not on $pbs(A, B)$, but on the side of $A$. $BQ$ will then certainly intersect with the perpendicular bisector, call the point of intersection $R$. $R$ is certainly not on line segment $AQ$. This is what you know and what you can use in your proof.

   a. Write down in $d$-notation what you need to prove.
   b. Since we have already proven part a, you do know something about point $R$. Note that as an equality.
   c. Also formulate an inequality, which contains $Q$, $R$ and $A$.
   d. Combine these to obtain the wanted conclusion.

18. Does the figure of the triangle inequality remind you of a problem we encountered earlier?
As a matter of fact, we also need to research the possibility that $d(A, Q) > d(B, Q)$. Of course $Q$ is on the other side of $B$, and this boils down the whole thing to consistently switching the letters $A$ and $B$. This is no longer interesting.

12. From exploration to logical structure

Introduction to this section

research
In the preceding we explored why the three Voronoi edges of three points (in general) meet in one point. We started off with the problem of partitioning an area
and then found out that a fundamental property of the concept of distance was of importance. The exploration developed as follows:

**Exploration path**

<table>
<thead>
<tr>
<th>From the question:</th>
<th>To the question:</th>
<th>To: Because eventually</th>
</tr>
</thead>
<tbody>
<tr>
<td>Why do three edges converge in one point?</td>
<td>Why is the Voronoi edge the straight perpendicular bisector?</td>
<td>the triangle inequality always holds</td>
</tr>
</tbody>
</table>

So far the exploration phase.

**logical structure**

Since we are reasoning, the logical structure will be the other way around when we look at it afterwards: first the triangle inequality, and deduct from there that the Voronoi edge and the perpendicular bisector coincide and then finally derive from there the statement about the concurrency of the three edges.

In this section we will repeat the whole in that last form. We will abandon the terminology of the Voronoi diagram; we will now make our proof mathematically pure, like this:

**Logical structure**

<table>
<thead>
<tr>
<th>From starting-point:</th>
<th>To:</th>
<th>And then to:</th>
</tr>
</thead>
<tbody>
<tr>
<td>The triangle inequality.</td>
<td>Properties of the perpendicular bisector.</td>
<td>Three perpendicular bisectors of a triangle through one point.</td>
</tr>
</tbody>
</table>

We will now formulate the most important statements, which will be proven concisely as theorems. We will number the theorems, as this will make future
reference easier. That is exactly what happens in a logical structure: what is proven before, you can use later.

**Starting-points: triangle inequality and Pythagoras**

The triangle inequality can be proven from other, more primitive starting-points. However, we will not do this. It will be our first theorem.

**Theorem 1** (Triangle inequality)

For each set of three points \( A, B \) and \( C \) holds: 
\[
 d(A, C) \leq d(A, B) + d(B, C).
\]
The equal sign occurs only if \( B \) lies on the line segment \( AC \).

In all other cases a real inequality occurs.

Next an exercise in practicing the use of the triangle inequality.

Four points are given: \( A, B, C \) and \( D \). \( P \) is the point of intersection of the line segments \( AC \) and \( BD \). \( Q \) is a point different from \( P \). Note: \( P \) is special, \( Q \) is arbitrary.

You have to show that

the four distances from \( Q \) to \( A, B, C \) and \( D \) together are bigger than

the four distances from \( P \) to \( A, B, C \) and \( D \) together.

19 a. a. Write down the to be proven statement using the \( d \)-notation as follows:

To show: \( d(P, A) + \ldots \leq \ldots \)

b. Then start the proof with:

Proof: ... and use (one or more times) the triangle inequality.

For the sake of completeness the Pythagorean theorem is stated below. You saw a proof of it when you did the extra exercises on page 69.

**Theorem 2** (Pythagoras)

If in a triangle \( ABC \) angle \( B \) is right, then this equality holds:

\[
 d(A, C)^2 = d(A, B)^2 + d(B, C)^2.
\]
Part I: Distances, edges & domains

*The perpendicular bisector*

We gave a definition of a perpendicular bisector. We will copy it here.

**Definition: Perpendicular Bisector**

The perpendicular bisector of line segment $AB$ is the line which passes through the midpoint of $AB$ and is perpendicular to $AB$.

The main properties of the perpendicular bisector are mentioned in the next theorem, which we showed before, on page 72:

**Theorem 3**

The perpendicular bisector of line segment $AB$ is the set of points $P$ for which hold $d(P, A) = d(P, B)$.

For points $P$ outside of the perpendicular bisector holds:

- If $d(P, A) < d(P, B)$, then $P$ lies on the $A$-side of $pbs(A, B)$.
- If $d(P, A) > d(P, B)$, then $P$ lies on the $B$-side of $pbs(A, B)$.

You can apply the theorem in the following problem. Think about this: if two points of a Voronoi edge are known, you know the entire edge.

20. In this delta wing $AB$ and $BC$ have equal length and also $AD$ and $DC$ are of the same length. Show that $BD$ is perpendicular to $AC$.

*Perpendicular bisectors in the triangle*

**Theorem 4**

In each triangle $ABC$ the perpendicular bisectors of the sides meet in one point.

21 a. In the section ‘An argumentation about three-countries-points and circles’ there was a problem: the three centers were not allowed to lie on one line. How did we get around that here? If the three points do lie on one line, what would happen to the three perpendicular bisectors? Sketch a (complete) figure.

b. How is the three points on one line situation excluded by the wording of the theorem?
Adjacent you see a figure, which illustrates the theorem. $M$ is the point of intersection of the perpendicular bisectors $AB$ and $BC$. On the next page you see a scheme, which represents the proof.

22. In fact, this is the proof as it was given in exercise 3, page 60.

a. Point out exactly how the different parts of the exercise correspond to the ones of the scheme.

b. Step 1 and 1-bis differ from the conclusion step. In which way?

**The circumscribed circle**

**Definition of the circumcircle**

The circumscribed circle of a triangle $ABC$ is the triangle's circumscribed circle, i.e. the circle that passes through each of the triangles’ three vertices $A$, $B$ and $C$.

The theorem about the circumcircle is no easy to formulate.
Theorem 5

Each triangle $ABC$ has one and only one circumcircle. The center of the circumcircle, i.e. the circumcenter, is the intersection of perpendicular bisectors of the triangle.

The proof is simple: the point $M$, where the three perpendicular bisectors intersect, has equal distance to each of the three vertices and is the only point with that property.

We will now do some exercises with circles and perpendicular bisectors.

23. This figure shows two circles with equal radii. Their centers are $A$ and $B$. Based on which theorem do the intersections $P$ and $Q$ lie on the perpendicular bisector of line segment $AB$?

This is a recipe to construct the perpendicular bisector with compass and ruler. Do not use a protractor or the numbers on the ruler.

24. Examine how Dürer used this technique for the front page of this chapter, when finding the circumcircle about the three points $a$, $b$ and $c$.

25. Pick out a few of the triangles on the next page and determine the intersection of the perpendicular bisectors and sketch the circumcircle. In at least one case, use the construction with compass and ruler.

Choose these in such a manner that one center lies inside, one center lies on and one center lies outside the involved triangle. How does this inside-on-outside link to the shape of the triangle?

26. Here a part of a circle is given. Determine, using only compass and ruler, the center of the circle. (Hint: to start, place some points on the circle)