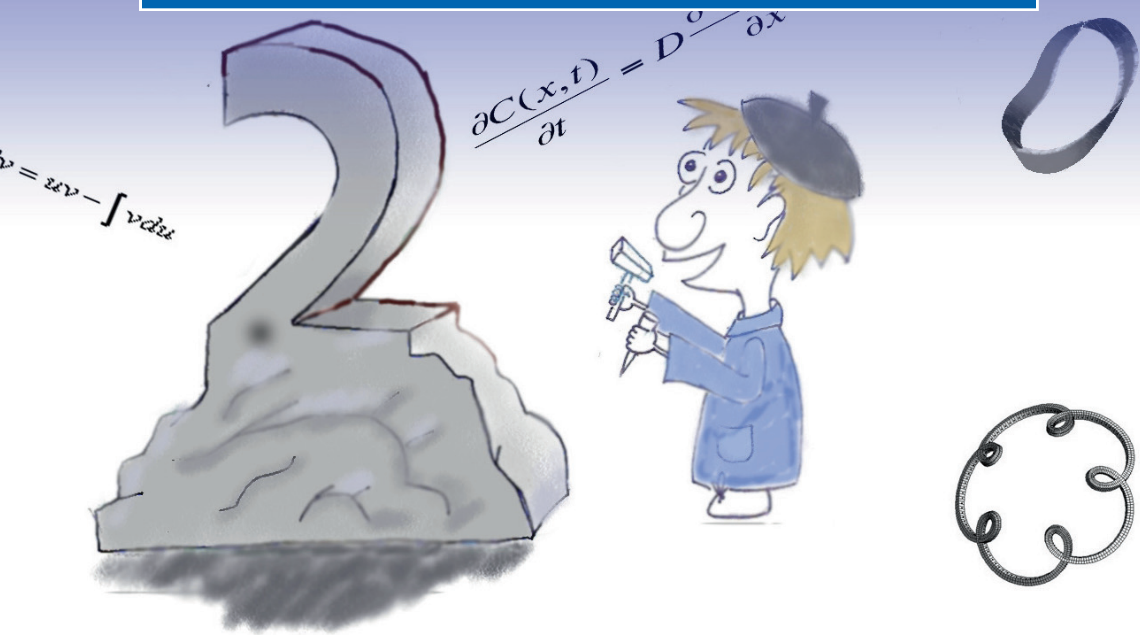
Creativity in Mathematics and the Education of Gifted Students

Edited by
Roza Leikin, Abraham Berman and Boris Koichu



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INTRODUCTION

Creativity and giftedness are underrepresented topics in mathematics education research. At the same time, mathematics is a rare context in psychological and educational research focused on creativity and giftedness. This book brings forward *mathematical* creativity and *mathematical* giftedness as important topics in educational research. This is a right time for doing this. On one hand, research on creativity in mathematics and on education of gifted students has recently grown up and developed a number of relatively stable conceptual frameworks, appropriate research methodologies, standards of communicating research results and of policy making. On the other hand, there is an apparent lack of shared understanding of mathematical creativity, giftedness and ways of their promotion within and across communities of mathematics educators, psychologists, educational researchers, mathematicians and policy makers.

The book is a result of a consolidated effort of a group of experts in the aforementioned fields. We aimed at addressing important contemporary challenges associated with the development of mathematical creativity in all students and with the realisation of mathematical talent in gifted students. The contributors of the book have different backgrounds and represent different standpoints on cognitive, affective and social factors of creativity and giftedness. Despite the wide range of the standpoints adopted by the authors, the following positions are shared throughout the volume:

- Prominent mathematicians are mathematically gifted and creative individuals. Systematic study of their experiences is an indispensible and irreplaceable source of knowledge about profound mathematical thinking, components of mathematical giftedness, characteristics of creative processes and products. We still need to know how to recognize creativity and giftedness in a person *before* he or she has a chance to demonstrate outstanding mathematical achievements (i.e., when in a kindergarten or in a school).
- Mathematically gifted and promising students need appropriate treatment, including specially designed learning environments, curricula and pedagogical approaches. The design of programs and educational frameworks for the mathematically gifted requires careful consideration of the complexities of the cognitive, affective, and social domains in light of contemporary research as well as careful consideration of mathematical content.
- Mathematical giftedness and mathematical creativity are interrelated though not identical concepts. Theoretical models that signify the relationship between general creativity, general giftedness, mathematical creativity and mathematical giftedness require thorough empirical validation.
- The diversity of definitions of mathematical creativity and giftedness should not be a too heavy obstacle for pedagogical matters. Despite the chosen definition, the development of mathematical creativity *in all students* is one of the major objectives of school mathematics education.

– Mathematical challenge is a core element of mathematics education for all students, but challenging the mathematically gifted is especially demanding educational task.
– Teachers must receive an appropriate training for supporting mathematically gifted and promising students and developing creativity in all students.

The chapters of the book are organized as follows:

– Perspectives on Creativity and Giftedness in Mathematics.
– Psychological Aspects of Creativity and Giftedness and Educational Policy.
– (Developing) Mathematical Creativity and Giftedness in Different Educational Contexts.
– Synthesis.

The reader may note that the first three parts are not mutually exclusive. While the titles of the parts adequately embrace the included chapters, some chapters might appear also in a different part of the book. This is because the chapters that present theoretical perspectives on creativity and giftedness delve into psychological and social aspects of the phenomena; the chapters that contain recommendations for policy making include examples from different educational and mathematical environments and contexts; the chapters on learning environments and contexts for supporting creativity and giftedness unavoidably contain elaborations of theoretical constructs in use. In addition, most of the chapters contain examples of challenging mathematical problems or classroom situations intended at developing mathematical creativity or giftedness.

The way of organizing the book we have chosen for the sake of coherency reflects just one way of understanding what "coherency" might mean in such a complex field as mathematical creativity and giftedness. Overall, we believe that the book presents a balanced set of papers that essentially reflects the state of the art in the field. The book also suggests an agenda for future research and development. We hope that the book will be interesting and useful for a broad audience of readers, including mathematicians, teachers' educators, mathematics teachers, educational researchers and psychologists.

We, the editors, would like to deeply thank Templeton Foundation for the generous support of the 5th International Conference "Mathematical Creativity and the Education of Gifted Students" and the International Workshop "Intercultural aspects of creativity: Challenges and barriers". These two events brought together the authors of the book in Haifa in February-March, 2008, and thus served as a starting point of the project. Our deep gratitude goes also to all the contributors.

Roza Leikin
Abraham Berman
Boris Koichu
Haifa
May 2009

PART I: PERSPECTIVES ON CREATIVITY AND GIFTEDNESS IN MATHEMATICS

AVI BERMAN AND GILAH LEDER

1. THE PLEASURE OF TEACHING THE GIFTED AND THE HONOR OF LEARNING FROM THEM

This chapter is based on a lecture by the first author and was written with the help of, and as an interview with, the second author.

Gilah: The title you have chosen for your talk, and now the paper based on it, has a wonderful implicit message: that working with bright young minds is both a privilege and a challenge. Can you tell me one or two of your favorite teaching experiences?

Avi: Sure! Let me give you two quite different examples.

Nir and Maria were high school students when I invited them to participate in a graduate course on Graphs and Matrices. They were very gifted students (and now are very gifted professors) but naturally did not have the background that the graduate students in the class had. The way we handled this difficulty was that after each class I met with the young students, covered briefly the unknown background material needed for this class and gave them the references they needed to master it. It worked.

Maria Chudnovsky

In the nineties I was in charge of the pre academic education program at the Technion. In this capacity I was asked by the Minister of Education to be in charge of a young gifted student, Gabi, who had asked to be exempted from the matriculation examination. Gabi was interested in Mathematics and did not want to "waste time" studying Physics. We had to discuss many

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episodes in the history of physics in order to convince him how important it is to understand physics for the development of a mathematician.

Gilah: What did you learn from working with students such as Nir, Maria and Gabi?

Avi: The main thing that stands out is “Yes. They can”. All the three students whom I mentioned above became mature and successful mathematicians. In particular, Maria Chudnovsky is now a professor at Colombia University and a world leader in Graph Theory. She knows much more than what I knew when I taught the graduate course mentioned above and what I know now. I want to emphasize that, in general, I would not recommend accelerating high school students to a graduate class even if they are highly gifted. I must admit that I did it for my own pleasure and because this was the course that I was teaching at that particular semester. What made it work were Nir’s and Maria’s strong motivation and persistence and the fact that they could work together. The moral of the episode about Gabi is that even for a mathematically gifted student education should not be limited to mathematics. (Gabi, by the way, is also a gifted musician).

Gilah: Can you briefly explain Maria’s claim to fame?

Avi: Her great achievement was solving one of the major outstanding open problems in Graph Theory – The Strong Perfect Graph Conjecture. Let me try to explain this conjecture (now theorem).

The chromatic number of a graph F is defined as the smallest number of colors that can be assigned to vertices of F in such a way that every two adjacent vertices receive two distinct colors; the clique number of F is defined as the largest number of pairwise adjacent vertices in F. An induced subgraph of a graph G is a subset of its vertices and all the edges of G connecting vertices in the subset. Trivially, the chromatic number of every graph is at least its clique number. A graph G is called perfect if, for each of its induced subgraphs F, the chromatic number of F equals its clique number.

A hole is a chordless cycle of length at least four; an antihole is the complement of such a cycle; holes and antiholes are odd or even according to the parity of their number of vertices. No odd hole is perfect (the clique number of an odd hole is 2 and its chromatic number is 3) and no odd antihole is perfect (the clique number of an antihole with $2k+1$ vertices is k and its chromatic number is $k+1$).

In 1960, Claude Berge conjectured that a graph is perfect if (and only if) it contains no odd hole and no odd antihole. This conjecture became known as the Strong Perfect Graph Conjecture*. (The perfect graph theorem, proved by Lovasz in 1972 says that a graph is perfect if and only if its complement is perfect). The conjecture was open for more than 40 years and was considered as one of the major open problems in Graph Theory. The conjecture was* *proved only in 2002 by Maria and Seymour (her Ph.D supervisor), Robertson and Thomas (see [Chudnovsky, Robertson, Seymour & Thomas, 2006]).*

Gilah: So far we discussed one way of working with gifted youngsters – mentoring. Can you describe other ways?

Avi: Two other venues are enrichment classes and research camps.

I recall the first enrichment class that I taught. It was offered to middle high school students who came from schools in Haifa and neighboring townships and were recommended for the class by their teachers.

Some 15 minutes into the class, one of the students (H) complained of a headache. That problem also occurred in subsequent lessons. Having had very little experience at that stage, it took me a few meetings to understand what was actually happening. I was used to asking questions and addressing them to all the students. Different ones replied and we discussed their solutions. When I thought that I understood the source of the headache problem, I specifically turned to H with some of the questions. He answered and, suddenly, there was no headache. What happened was that H who was one of the best students in his school class, was shocked to see that other students could answer a question before he understood it. Needless to say, this episode improved my sensitivity to students, gifted and ordinary. Incidentally, H became a successful engineer.

By the way, another student in that class, Dorit, became one of the world leaders in the study of Quantum Computing. She is now a professor at the Department of Computer Science and Engineering at the Hebrew University and was chosen by the prestigious and influential journal *Nature* as one of four outstanding young theorists to be profiled in the first issue of 2005. This issue commemorated the 100 year anniversary of Albert Einstein’s “wonderful year” in which he published four of his ground-breaking publications at the age of 26.

Dorit Aharonov

Gilah: You also mentioned research camps.

Avi: Yes. Each summer the Technion invites gifted boys and girls from all over the world to a research camp. The young students take part in real research at the Technion labs, assisting professors and their graduate students (see [Berman, Goldberg & Koichu, 2005]). The evenings are devoted to social activities and in one such evening the students were asked to describe some of their favourite problems. Here is one problem that I recall.

A prince has to choose one of the three King's daughters. One is the oldest, one is the youngest but they all look alike. The oldest is a liar. The young one speaks the truth. The middle one kills whoever marries her. The prince HAS to choose one girl. He can approach ONE girl and ask her ONE question. What shall the prince do?

Gilah: Lovely problem. Giftedness and problem solving are of course related.

I guess it is difficult to give a ready made, one-fits-all recipe for nurturing mathematical talent and deciding when acceleration is appropriate. What stands out for me in the examples of the successful students you have mentioned are the careful identification, in terms of both mathematical ability and personal characteristics (for example, curiosity, strong motivation, and a willingness to persevere), access to a supportive and rich intellectual climate, some administrative flexibility, and last but not least an enthusiastic and mathematically gifted teacher.

Avi: I agree that enthusiasm, flexibility and sensitivity are very important.

Gilah: You and your colleagues have explored the ways in which students solve problems: the heuristics which are involved in successful problem solving ([Koichu, Berman & Moore, 2007]) and the extent to which gifted students' conceptions of *effectiveness* and *elegancy* of a solution to a problem guide their problem solving behaviors ([Koichu & Berman, 2005]).

Terry Tao, a precocious young mathematician and a recent Field Medalist, wrote a book on problem solving (Tao, 1992, pp. 79-83) while still a teenager. He listed his strategies for problem solving as follows:

– Understand the problem
– Understand the data
– Understand the objective
– Write down what you know in the notation selected; draw a diagram
– Consider a special case; solve a smaller version of the problem; generalize the problem…
– Experiment with the problem; reverse the problem
– Simplify, exploit data, and reach tactical goals.

I know it is difficult to generalize, but do you recognize these tactics or see any commonalities in the way that Maria, Dorit and other outstanding students you have taught, typically approach mathematical tasks you set them or they set themselves?

Avi: You are right that it is difficult to generalize. In fact, when I taught Maria and Dorit they were in a very early stage of their career, before they solved the big problems. I did however observe all these tactics, as well as the heuristics suggested by Polya in his classical book ([Polya 1945]) in problem solving seminars that I gave at the Technion. The first three tactics are particularly useful in solving the 3 princesses problem mentioned earlier. At the Technion we have a program for exceptionally gifted students and I used this problem when interviewing candidates to the program, paying attention to how they understand the problem.

It is interesting that you mentioned Terry Tao. Recently (March 2009) he gave a special lecture on random matrices in the Technion. It was a great talk. Moreover, Nitsa Movshovitz-Hadar and I have a doctoral student, Batya Amit, who works on interweaving mathematical news into the school curriculum. After the talk Terry talked with Batya and suggested wonderful news flashes.

Gilah: So far you have talked from the perspective of the teacher. What about your own reflections as a student? What were your experiences as a student? Are there any teachers who stood out for you, who challenged you, or even provoked you – and if so, why and how?

Avi: I was lucky to have excellent teachers both in high school and at the Technion.

My Math and Science teachers in high school were professors at the Technion and the University of Haifa. This gave us an opportunity to study topics that are not part of the regular high school curriculum. I now try to do similar things in special high school math classes in which the Technion is involved.

Even in middle school I had exceptional teachers. One of them enjoyed presenting us with very challenging problems. For me working and sometimes solving these problems was a source of great enjoyment. For some of my friends it had a terrible effect, resulting in their leaving mathematics even at that early stage.

This reminds me of something that happened later, in high school. I wrote a little paper in which I proved a result in Set Theory and sent it to one of the leading experts in set theory, Prof. Abraham Fraenkel. He replied, telling me that the result was known, but he did it very gently, encouraging me to continue my interest in mathematics.

Gilah: I can sense that – after all this time – you are still grateful that Professor Fraenkel took the nascent research efforts of a young high school student seriously. How many others in his position would have bothered? And in ignoring such youthful enthusiasm, how many budding mathematicians are discouraged and turn instead to other fields?

Abraham Fraenkel

Avi: Most of my teachers at the Technion were great and challenging teachers. I will skip their names so that no one will be omitted but I want to mention several visiting professors who had a strong influence on my inclinations and future study and research. They included Alan Hoffman, Max Shiffer and Paul Erdos.

Alan encouraged my interest in Matrix Theory and taught me how to decide if a paper should be submitted for publication, (Menachem) Max influenced my interest in Applied Mathematics and created my appreciation of how the learning of mathematics should and could be motivated. Paul strengthened my love of problem solving and my interest in Graph Theory.

Gilah: Educators and policy makers in many countries have tried to identify what, in particular, promotes student learning generally or in mathematics specifically. In a few short sentences, through the examples you have given, you have come to the same conclusion usually reached after much time, effort, and money has been spent: the teacher. Teachers can and do make a difference.

Before we finish, I have one more question. What do you now know, and practice, about challenging gifted students "to give of their best" that you did not know early in your career? What is the best advice you can give to young mathematics and mathematics education professors?

Avi: The best advice for both students and teachers is to enjoy mathematics, but of course gifted students and gifted teachers know it and do not need my advice. This is a good time for a final word. I do not claim that Dorit's breakthroughs in Quantum Computing are the results of her taking an enrichment class as I do not claim that Maria's solution of the Perfect Graph Conjecture is due to the course that she took at the Technion when she was in high school. The points that I wanted to make are that I am proud of

Paul Erdos

taking part in their education and that of many other gifted students. I am also very grateful for the opportunity of learning from very gifted teachers.

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ALEXANDER KARP

2. TEACHING THE MATHEMATICALLY GIFTED: AN ATTEMPT AT A HISTORICAL ANALYSIS

This chapter attempts to sketch the history of teaching the mathematically gifted. The author's premise is that genuine education of the mathematically gifted is inextricably linked to the principle of equity in education because only by acknowledging social equality among all students can sufficient attention be paid to differences in their abilities. Consequently, the article deals mainly with the 20th century and focuses on two "case studies" — the United States and Russia (USSR) — while discussing certain tendencies in the development of education and certain concrete methods and organizational structures in education.

INTRODUCTION

This chapter is devoted to the history of teaching the mathematically gifted. The concept of mathematical giftedness still lacks a clear definition in the scholarly literature — researchers continue to argue about the relation between nature and nurture, nor is it always easy to distinguish mathematical talent from general abilities or from even broader characteristics, such as an interest in the subject matter (see, for example, the surveys of existing literature in Sheffield, 1999 and in Sriraman, 2008). The task of the historian is accordingly even more difficult, because we know far less about what went on in the classrooms a hundred years ago than we do about what goes on in today's classrooms, while insufficient attention to social and cultural differences between society as it is now and society as it was then can lead to superficial and inaccurate conclusions. It is therefore necessary to state at the outset that this chapter is intended to draw attention to a research topic rather than offer an exhaustive presentation of it. The aim of the chapter is to identify certain broad trends that unfolded in the history of education, and the traditions and methods of education that it comprised. Because of space limitations, we are constrained to focus only on certain episodes and certain countries. When talking about the history of the 20th century, we confine ourselves mainly to developments in the United States and Russia (USSR). These countries played important roles in both mathematical research and mathematics education, and they exerted a considerable influence on many other countries. To repeat, a detailed history of teaching the mathematically gifted worldwide still remains to be written.

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THE PREHISTORY OF TEACHING THE MATHEMATICALLY GIFTED

Reading a biography of a mathematician written in antiquity, we are unlikely to come across anything having to do with the manifestation of his mathematical gift in childhood. Diogenes Laertius writes about Pythagoras, for example, that he was the first man to feed wrestlers with meat, that he forbade his disciples to touch white roosters and to pick up what fell from the table, that he enjoined them to abstain from eating beans, and even that he considered the circle to be the most beautiful of all plane figures. But Laertius tells us nothing about how Pythagoras's mathematical talent developed or how he taught mathematics to his disciples, to whom he apparently left such precise and detailed instructions concerning so many other matters (Diogenes Laertius, 1972). Pythagoras's other biographers occasionally give slightly more detailed accounts of his education; for example, Porphyry reports that, in his childhood, Pythagoras studied with a lute-player, a wrestler, and an artist, and that he later went to Anaximander at Miletus to learn geometry and astronomy (Porphyry, 1919). But here, too, we hear nothing about his mathematical talent or about any special treatment he may have received in the course of his education.

Meanwhile, the concept of giftedness itself did exist in ancient times, of course, and it is natural to look to antiquity for one of the origins of our interest in the gifted (Tannenbaum, 2000). A person whom we would describe as talented was considered by the ancients to have been specially chosen by the gods (see, for example, Rabinovitch, 1985). This chosenness turned out to be, to a certain extent, the result of the person's education, and manifested itself in a correct approach to constructing his life. This is why the philosopher needed to embrace all manner of privations, such as the aforementioned abstention from beans, so that he may resemble a god by demonstrating his gift. But this gift was never regarded as a specifically mathematical endowment.

To be sure, the role of mathematics in education was considered significant. Plato's views on the crucial role of mathematics in the education of the philosopher are well known (Plato, 1991). Jaeger (1939-1945) noted, however, that in Socrates's time, the idea that the education of young men should involve the study of various sciences was something new, and that along with Plato's views on the matter, a contrary view also existed. According to this view, Socrates himself, although he recognized the necessity of a certain familiarity with mathematics, did not believe that the student should study it in too great a depth.

An education that passed from studies with the lute-player and wrestler to studies with the geometer, such as the one described by the aforementioned Porphyry, was quite typical (Boyd, 1965), except for the fact that not all students reached the geometer, and that the share of those who engaged in a serious study of mathematics, as far as can be judged, declined over time. Writing about relatively late antiquity, Marrou (1956) poses the question: "Who was supposed to be taught mathematics — everybody or a specialist elite? Did it form part of secondary school education — as it was supposed to, in theory — or was it entirely for the more advanced students?" (p. 183). His answer is that mathematics was studied seriously only by the elite, those who were preparing to become philosophers, while the vast majority of those who received an education stopped at subjects that would today be characterized as humanities.

To a certain extent, therefore, all mathematics education in antiquity that went beyond basic levels — mainly elementary arithmetic — may be viewed as a more advanced or selective form of education (note, also, that education was quite individualized: methods for working with an entire classroom of students evolved over the course of centuries, if not millennia). Its selectiveness, however, was based on the overall goal of a student's education (the study of philosophy) rather than on a student's exceptional performance. For this reason, Marrou concludes that the study of mathematics, at least in late antiquity, was pursued beyond the bounds of secondary education (to use modern terminology) but in the context of higher education. The selected elite consisted of future specialists rather than of children with the greatest aptitude for mathematics, although it is also natural to suppose that students who were more gifted mathematically and interested in the subject would have more frequently decided to study philosophy, all other things being equal.

It turned out, however, that it was difficult for students to become interested in mathematics, just as it was difficult for them to become aware of their own giftedness and just as it was difficult for anybody else to recognize their gift. As Marrou (1956) goes on to conclude:

> The people interested in the subject — mathematical specialists, or philosophers for whom mathematics was an indispensable preliminary — got no recruits from the secondary schools... mathematics no longer [had] any real place in the common culture. (pp. 183-185)

All this was even more true of the Middle Ages. Ridder-Symoens (2004) notes that, "since the natural sciences hardly had any place in the schools, individual researchers outside the schools, on the fringes or in the courts of princes and bishops, studied the translated texts that came to Europe during the twelfth and thirteenth centuries" (p. 119). These texts (including Euclid and Ptolemy) gradually, and not without difficulties, penetrated into the universities (see, for example, Leff, 1968), but both before and after this development, mathematics was still taught and studied largely outside the universities (not to mention educational institutions with more elementary programs). Centers of study sprung up, supported by the courts, and in time even by various guilds (Ridder-Symoens, 2004). Of course, these were not centers of mathematics education *per se* in the modern sense of the term. Both the nobles and the guilds needed specialists or engineers, as they are called today. It was at such centers that the mathematicians of the time often worked, and it was there, too, that in many ways they received their education.

In medieval times, only a small part of the population received an education at all. Based on his studies of French colleges, however, Gillispie (2004) makes the following paradoxical remark: "The secondary educational system provided a much broader social exposure and rather more opportunity for rising in the world on the eve of the Revolution than it was to do in the bourgeois France of a hundred years later, though less than it had done a century and more earlier" (p. 133). The medieval school recognized “the necessity of taking into account the individual talents of a child” (Ridder-Symoens, 2004, p. 123), and a talented person who started out low on the social ladder (even if not at the very bottom) could make substantial progress upward. At the same time, it is fair to say that only those people devoted themselves to the serious study of mathematics who had an aptitude for it. The need for mathematically gifted individuals, however, was so slight that there was no question of specialized education for them on a wide scale, nor was there any awareness that such a group of mathematically gifted individuals even existed. Although today we recognize the naiveté and erroneousness of an image of the Middle Ages as a period “totally lacking” in observation, invention, and innovation, it is clear that it was a profoundly “anti-mathematical” era, an age of imprecision and approximation (Koyré, 1961).

It was during the 16th-17th centuries that a substantially greater number of significant mathematicians appeared than had been active previously. This was a period that saw the formation of capitalist society, with its love of accounting and recognition of the possibilities of “profiting by using spiritual abilities” (Sombart, 1913). The idea that creativity should be encouraged was in the air, but it would be interesting to study systematically the exact type of education that enabled mathematicians of the time to achieve their successes. Looking at the best known biographies of the leading mathematicians of the 16th-17th centuries, it is difficult to see any common pattern in their education apart from the large role played by independent study. Aries (1992), for example, describes how a student’s giftedness could affect the time it took to educate him; but he also notes that the age at which it was considered “normal” for a child to be a student differed in the16th-17th centuries from what we would now consider it to be, so that it would be misleading to assume that the educators of the time sought to establish educational acceleration for the gifted as a matter of principle.

The growing need for specialists led to the expansion of the network of specialized schools and to improved methods of teaching and selecting students for such schools. For example, by 1697, entrance exams in mathematics had been introduced in the French Corps of Military Engineers (Belhoste & Chatzis, 2007). Such military engineering schools, however, remained accessible only to the nobility, and, no less important for our purposes, they remained narrowly specialized institutions. The French Revolution took a significant step when it simultaneously changed both of these factors, leading, as Belhoste (2003) argues, to the creation of the French technocratic class and exerting a substantial influence on the teaching of mathematics around the world, including the teaching of mathematics to the mathematically gifted.

The striving for equality, conceived of as an absence of barriers between social classes, led in particular to the elimination of the division between civilian and military engineers, the latter having previously occupied a more privileged position. At least in part thanks to this development, the *Grandes écoles* that appeared in France were in large measure oriented toward general scientific, and above all mathematical education for highly gifted and well-prepared students. This in turn necessitated reforms at the preceding stages of education as well (Belhoste, 2001; Gillispie, 2004; Zwerling, 1990).

Belhoste and Chatzis (2007) describe the first rungs of the system that gradually developed as follows:

1. Secondary education was provided by the *lycées*, the new secondary level schools created in 1802, where students learned Latin and mathematics.
2. One-year preparation (in sciences) was conducted in specific *Mathématiques spéciales* classes in the *lycées*.
3. An entrance examination was administered for the École Polytechnique, whose program focused mainly on mathematics. Each year, examiners travelled around the country to test candidates (p. 214).

We would not yet characterize this system as a system of education for the mathematically gifted, but an important step had been taken toward its creation.

DISCUSSION: TERMINOLOGICAL ISSUES

The education of the mathematically gifted has so often been criticized (although unjustly, in our opinion) as a form of elitism that sometimes truly elitist education — which is what education was for almost all of mankind's history – appears in a certain sense to have been education of the gifted. For centuries and even millennia, all schools were schools for the select few. Against the background of such schools, educational institutions with more advanced programs were extremely demanding of the students, which led to the development of traditions that remain important to this day. For example, many of the problems that are given to mathematically gifted schoolchildren today go back to problems that were first assigned in schools in antiquity or during the 17th century. Mark Twain was engaging in satirical exaggeration when, in his novel *A Connecticut Yankee in King Arthur's Court,* he made the knights admit to the academy not the capable and educated candidates who had been prepped by the protagonist, but individuals who were illiterate but wellborn. In reality, no illiterate aristocrat took any advanced scientific courses in anything; there was no need for him to do so. Indeed, these courses were not attended exclusively by the wellborn, and their difficulty guaranteed that only talented individuals would be able to complete them successfully.

However, considering the fact that even for the most gifted representatives of the lowest social classes advanced courses remained inaccessible simply because they had no opportunity to manifest their giftedness, it is clear that the schools of antiquity or the Middle Ages cannot be characterized as schools for the gifted. The principle on which they were based was different. Only the recognition of students' equity made it possible to take fully into account differences in their individual abilities and performance.

It is only relatively recently that mathematical giftedness began to be perceived as a distinctive phenomenon (see Krutetskii, 1976, for a review of the literature). Many aspects of the French educational system described above, which took shape after the Revolution and Napoleon, may have given it some resemblance to a system for the education of the mathematically gifted. But the recognition that it is preferable to teach mathematically gifted students in a different way from other students has emerged much later (see, for example, Frechet, 1925, for whom, however, the "correct" form of education for the gifted was equivalent to the general education that had existed earlier).

In the present chapter, the education of the mathematically gifted is understood as part of a general, rather than a professional education. The problem of the mathematical education of schoolchildren who have displayed exceptional mathematical abilities is not identical, in our view, with the problem of the preparation of specialists in a particular field, for which the students must receive special training in school, but these two problems are connected. Thus, today the education of the mathematically gifted is often cultivated in order to produce much-needed specialists. Conversely, in the history of mathematics education we often come across programs for the preparation of specialists, which have left to the education of the mathematically gifted pedagogical traditions and methodological materials, but which were narrower than general education in their design and their aims, and consequently involved a smaller number of students.

A genuine history of the education of the mathematically gifted presupposes, together with a recognition of students' equity, a recognition of the distinctive nature of mathematical giftedness and an awareness of the desirability of a special approach to teaching mathematically talented students as part of a general school-level education. This is something that we encounter only in the 20th century.

A CASE STUDY OF THE UNITED STATES

The first attempts to obtain a precise measure of students' relative abilities were made in Europe, where at the beginning of the 20th century Alfred Binet began to produce intelligence tests. But probably nowhere was the use of these tests as widespread as in the United States, where for a certain time they became an educational tool of the utmost importance. Prominent psychologists such as Edward L. Thorndike and Lewis Terman were among those who supported and developed intelligence tests. Educational historian Diane Ravitch (2000) points out that "despite the debates of the 1920s, group intelligence testing became a routine feature in American public schools and even in many private schools" (p. 155). Based on the results of such testing, students in middle and high schools were divided into homogeneous groups. Educators had such great faith in the accuracy of the evaluation of each student's intellectual potential that, starting in the 1920s, even college admissions came to be determined largely not by exams that tested what students had learned in one or another academic subject, but by tests that determined the students' general capacity to learn — scholastic aptitude tests. As Ravitch (2000) notes:

> The intelligence testers promoted fatalism, a rueful acceptance that achievement in school is the result of innate ability, not sustained effort by teachers and students. The cult of IQ became an all-purpose rationale for students' lack of effort and poor teaching. (p. 161)

But while differences between students' general abilities were incessantly emphasized and well recognized, the importance of specific subject matter, not students' general aptitude but their aptitude for a specific subject, remained insufficiently appreciated. Naturally, this does not mean that there was a complete absence of special work with students who displayed an interest specifically in mathematics. In general, those who had more aptitude for mathematics could take more advanced courses. The fact that students were divided into homogeneous groups according to ability facilitated this, and was seen as a primary means of providing for individual differences (see, for example, Davis's textbook for teachers, 1951, which was written much later but expressed the same views).

In an earlier book for teachers, which went through numerous editions, Young (1925) recommends establishing mathematics clubs for upperclassmen interested in mathematics. The author carefully notes that in the clubs with which he is familiar, "club membership was restricted to pupils of the last school year" (p. 164). The author suggests that, during club meetings, students should discuss in greater depth what they have covered in class; that they should offer alternative proofs for the assertions proven in class; and even that they should discuss topics not included in the normal curriculum, such as cubic equations. Nor does the author fail to mention topics from the history of mathematics and recreational mathematics (he pays particular attention to card tricks). Similar recommendations are given by another popular textbook for the preparation of teachers, written by Breslich (1930).

Thus, educators worked with gifted students within the school since the school could become a venue for various types of special classes. But such classes could also fail to materialize, in which case students had to do with ordinary courses, perhaps being able to complete them more quickly than their schoolmates. Much later, Passow (1956) argued that "nor should there be raised a false conflict between education for all children and special provisions for the education of the talented." The very need to explain this point, however, means that such a conflict did frequently arise in practice. Tannenbaum (2000) notes that the interest in educating the gifted reached several peaks in the history of education in the United States, and that between these peaks there was a lack of interest in this matter. Tannenbaum concludes, not without irony, that American society is never interested in teaching both the most and the least successful achievers at the same time.

During the 1950s-1960s, when the fundamental changes taking place in science — and, what is perhaps even more important, in its role in the life of society — started to be understood, the interest in teaching the mathematically gifted reached an unquestionable peak. Significantly, the rise of this interest antedated Sputnik (1957). One example of it was the program described by Brinkman (1954), involving the development of advanced courses in mathematics for schools, with the participation of several colleges and schools.

In 1957, the National Council of Teachers of Mathematics (NCTM) published a brief pamphlet on teaching the mathematically gifted (Vance, 1957). It contained a list of topics that may be used for the enrichment of ordinary courses. The pamphlet recommended that such topics be used, for example, for individual and group projects. The pamphlet also described an experiment with acceleration, in which a three-year curriculum was covered in two years. It also noted the problem of providing support for gifted students in schools where it was impossible to create a special, separate class for them. It was suggested that in identifying gifted students, educators do not confine themselves to looking at their scores on aptitude and achievement tests, but, for example, note who is “sincerely interested in mathematics,” who is not satisfied “with a minimum discussion but ask[s] questions,” etc. (p. 20).

In his influential report, James B. Conant (1964) emphasized the necessity of grouping by ability in every subject, pointing out that a student may be in the top group in one subject and in a middle group in another. He proposed special programs of study for the academically talented (approximately 15% of the entire student population) and for the highly gifted (approximately 3%). The latter were to be identified in the seventh and eighth grades, with a view to allowing them to take advanced placement classes (i.e., college-level courses) during their last year of high school.

The program, again, was to be implemented inside the school (“the *comprehensive* school”) with a certain degree of support from colleges. Other approaches, however, were also developed. Honors classes offered by individual schools “eventually evolved into special high schools” (Vance, 1957, p. 9). The earliest among these was the Bronx High School of Science, which was established in 1938. Later, shortly before and especially after World War II, many other specialized schools emerged, giving rise to one of the important forms and traditions of working with the mathematically gifted (Vogeli, 1997).

These schools, which had assimilated in part the European experience, were not oriented exclusively toward mathematics, not only because, like all American schools, they provided instruction in other subjects, but also because the students were generally not expected to choose mathematics as their future profession. Nevertheless, these institutions offered substantially expanded opportunities for the mathematically gifted.

The educational models employed by these schools varied widely. Simplifying somewhat, we can say that some of these institutions considered their central task to be the creation of additional opportunities for an *accelerated* study of mathematics, while others aimed at the *enrichment* of such study. The Texas Academy, for example, was effectively made part of a college, so that students could learn in university lecture halls and laboratories, often under the supervision of university professors, and the classes they attended were in essence at college level (Stanley, 1991). By contrast, the North Carolina School of Science and Mathematics and the Illinois Mathematics and Science Academy (IMSA) offered their students many courses that differed fundamentally both from the school-level curriculum and from the college sequence. One example is the IMSA’s course on “Mathematical Investigations,” which, according to the IMSA mathematics course information sheet, was offered in conjunction with courses in the sciences. Although the course also covered largely the traditional contents of such courses as algebra and precalculus, it was oriented much more toward data analysis, modelling, and applications.

Gradually, a country-wide system of mathematics competitions evolved in the United States, which came to play an important role in the education of the mathematically gifted. For example, in New York City, interscholastic competitions in mathematics started being held as early as 1913. After 1919, from 9 to 16 schools participated in competitions organized by the New York City Interscholastic Mathematics League for many decades. In 1934, the New York University Chapter of Pi Mu Epsilon began conducting annual mathematics competitions, and in 1950 the Mathematical Association of America began organizing competitions for schoolchildren as well. Consequently, mathematics teams began to form in schools, and special classes started to be offered for their members. Vance (1957) lists some of the topics covered in these classes: indeterminate equations, partial and continued fractions, vectors used to solve geometric problems, constructions by algebraic analysis, symbolic logic, and Boolean algebra (p. 12). Over the years, the number of topics discussed in these classes grew considerably. Because these classes were oriented toward participation in competitions, they devoted particular attention to problem solving.

Yet another approach to working with the mathematically gifted consisted in organizing summer and weekend schools and classes. These classes were held usually at universities (see, for example, Tannenbaum, 2000, on the experience of Teachers College, Columbia University; Eberle, 1971, or the website of the Ross program, founded by Arnold Ross in 1957, at http://www.math.ohio-state.edu/ross/). Since we cannot list all these programs here, we focus briefly on a single example: the Study of Mathematically Precocious Youth, founded by the outstanding psychologist Julian Stanley at Johns Hopkins University in 1971 (Stanley et al., 1974, 1977).

The program was based on identifying gifted students by means of scholastic aptitude tests. The selected students were offered special courses that were oriented mainly toward the acceleration of their education in mathematics, but to a certain extent also toward its enrichment, as among the courses offered were some that went beyond the bounds of the normal sequence and included such topics as number theory. Muratori et al. (2006) brings vivid examples of the program's success: two young men, whose talents manifested themselves at a very early age, were identified by the program and went on to amass an impressive collection of awards of all types (including medals at the International Olympiad). Eventually they became successful professors of mathematics. Whereas one of them skipped classes and obtained his Ph.D. at age 21, the other skipped only one class because his parents considered advancing too rapidly to be hazardous for his emotional and social development (although the first child's parents were also concerned about their son's social development). Thus, the program was highly diversified and individualized, aiming at providing assistance (including psychological support) to gifted children on the path that together with their parents they chose for themselves.

Overall, the American system of working with the mathematically gifted is greatly varied. Naturally, it is difficult or even impossible to prove experimentally that one approach is more fruitful than others. After all, the success of a graduate from one or another school or study program can hardly be taken as an indication of the success of that particular school or study program: who knows what would have happened if this student had been taught differently? Maybe he or she would have performed even better.

It is clear, however, that existing programs are often perceived as insufficient. Julian Stanley (1991) writes about the boredom that a gifted schoolchild feels in the mathematics classroom, a boredom that leaves traces that cannot be erased. The title of an old article, “Let’s Do Something for the Gifted in Mathematics” (Johnson, 1953), still sounds relevant today, when many gifted schoolchildren remain without access to any of the existing programs, especially if they come from socio-economically or historically disadvantaged portions of the population.

A CASE STUDY OF RUSSIA (SOVIET UNION)

Russian education, developed in the 18th and 19th centuries based on foreign models, was initially oriented toward a very small number of students (Polyakova, 1997). By the second half of the 19th century, the recognition of differences between the needs of students oriented toward the humanities and students oriented toward technical disciplines led to the creation of German-style “Real-schools,” with a highly intensive program in mathematics (Ganelin, 1954). Yet there was no appreciation of the specific nature of mathematical giftedness, and more broadly, of the specific nature of the mathematical way of thinking, at this time.

One of the first Russian studies to analyze the nature of mathematical giftedness was written by Mordukhay-Boltovskii (1908). Noting that mathematical abilities cannot be reduced to diligence and a good memory, the author lists several specific traits, such as imagination, the capacity to think quickly, and what he calls cleverness. He draws no conclusions about any special education for the mathematically gifted; on the contrary, he expresses the opinion that the educational and developmental value of studying mathematics is limited. In his view, this value derives entirely from the fact that mathematics is the only subject in which the student thinks independently.

After the revolution of 1917, the government started using all the powers of the state apparatus to promote what were considered innovative educational theories, including those that advocated testing students to determine the coefficient of their intellectual development and taking this coefficient into account in their subsequent education. The subject matter often turned out to be unimportant in pedagogical discussions and the organization of education at this time (Karp, 2009); children with a low level of development were generally sent to special schools, and children with a high level of development were given the opportunity to display their talents within a collective and to study individually. At the end of the 1920s and the beginning of the 1930s, when the government started restructuring the country along more militarized lines, and the need for engineers arose on a massive scale, intelligence testing was terminated. Zhdanov, a Communist Party ideologue, described the questions that psychologists had used during the previous decade as “inquisitorial” and explained the motivation for the creation of the theories that had been previously popular as follows:

> For what purpose does the bourgeoisie need such a science? It needs it in order to separate rich children from the children who embody the developmental defects of the capitalist system. This theory was created by the bourgeoisie in order to prevent these defects of capitalist society from creeping into bourgeois schools, in order to prevent them from influencing these more mentally gifted children. This theory has created a bourgeois aristocracy[1]. (Stenogramma, 1936, p. 20)

Subject matter-oriented teaching, which had only recently been criticized, returned to the schools. Mathematics, the foundation of technical knowledge, became the focus of heightened attention for the entire period of schooling (Karp, 2009). At this point, the state also needed research mathematicians to advise on improving mathematics education. At a meeting called by the people’s commissar (minister) for education, the prominent St. Petersburg mathematician, Grigory Fikhtengolts, remarked: “In our school, mindless routines are the rule in problems and in methodology. Students are not taught how to take an independent approach” (Soveschanie, 1936, p. 58). The issue being discussed was how to improve education for all students, but naturally research mathematicians were particularly concerned about those who were mathematically gifted.

In 1934, the first mathematics Olympiad in the USSR was held in Leningrad. Its aim was not so much to select talented students (there were no further rounds in this competition) as to attract students to mathematics. Famous mathematicians spoke with the winners and gave them a small collection of mathematical books (Fomin, 1994). Almost simultaneously with the creation of the Olympiads, the “Popular Lectures in Mathematics” series, intended specifically for schoolchildren, began to appear (Boltyanskii and Yaglom, 1965). A little earlier, a system of so called “circles” (clubs) has been established for those interested in mathematics. In 1933, the so-called scientific station for gifted schoolchildren was established in Leningrad (Fomin, 1994). After the Moscow Olympiad for schoolchildren (1935), mathematics circles from several schools were transferred to Moscow State University, forming new circles that subsequently exerted a massive influence on mathematics education (Boltyanskii and Yaglom, 1965).

Initially, students in the mathematics circles listened to prepared reports under the supervision of circle leaders. Subsequently, however, problem solving became the principal form of activity, often preceded by a lecture by the circle leader on the essence of the topic being studied. The circle leader guided the students, selected (wrote) the problems, and evaluated the solutions. The important role played by the circle leader can be seen in the fact that the authors of Shkliarskii et al. (1962) listed David Shkliarskii as the first author although he had died many years before the publication of this book. Shkliarskii used to be their circle leader and had in many respects reformed the work of Moscow's mathematics circles. The most prominent mathematicians served as circle leaders and organizers of Olympiads. The involvement of scientists with the schoolchildren and teaching through problem solving were the seminal features of the Russian system of education for the gifted.

A crucial stage in the education of the gifted was the creation of specialized schools (Chubarnikov & Pyryt, 1993; Grigorenko & Clinkenbeart, 1994; Donoghue et al.; Vogeli, 1997). In the late 1950s and early 1960s, Soviet education went through another series of reforms aimed at ensuring that students acquired some kind of specialization in school. Mathematicians and mathematics educators identified an additional opportunity here. In 1959, the first classes of computer programmers appeared (in Moscow's school No. 425, Shvartsburd, 1963). In 1963, boarding schools specializing in physics and mathematics were established under the aegis of leading universities.

The most prominent scientists in the Soviet Union participated in the creation of these schools. The mathematician Andrey Kolmogorov founded a boarding school specializing in mathematics in Moscow. The physicist Isaak Kikoin, who in the 1970s became the first editor-in-chief of *Kvant*, a magazine of physics and mathematics for schoolchildren, also played an important role during the preceding stage in the creation of specialized schools. The influence of mathematicians and physicists, the creators of the USSR's "nuclear shield," on the leadership of the country was extremely important. The Soviet government saw the specialized schools first and foremost as institutions for the formation of the future employees of the military-industrial complex.

The specialized schools were not special in the sense of neglecting the study of other — not physical or mathematical — disciplines. All other subjects were taught based on the same curricula that were used in other schools in the country, but substantially more time was devoted to the study of mathematics, and the mathematics curriculum was significantly more challenging. The mathematics curriculum was greatly expanded (students were taught many topics in analysis, modern algebra, discrete mathematics, non-traditional topics in geometry, and so on), and even more important, the way in which these subjects were studied was significantly different.

For example, under the supervision of N. N. Konstantinov and others (Gerver et al., 1967), several Moscow schools developed a system of teaching in which the teacher practically never told the students the "theory" behind the subject being studied: the students would derive the entire theory themselves. They would be given sheets of paper with basic definitions and problems; they would then solve these problems, at their own speed, and hand them in to the teacher and the assistants. In this way, the entire course was taught through problems.

Many forms of activity not employed in ordinary schools, or employed differently and to a far lesser extent, became widespread in schools that specialized in mathematics. Long-term assignments were commonplace and included both problem sets and various types of projects. Oral exams often played an important role in these schools. These exams were not so much formal examinations of the students' knowledge as problem solving sessions that lasted several hours and put the students in contact with the more knowledgeable person administering the exam (as a rule, graduates of the schools were invited to administer the exams). Another key feature of the system was the great variety of mathematical competitions (Karp, 1992).

Specialized schools went through several stages of development (Donoghue, et al., 2000; Karp, 2005). In the 1970s and 1980s, during the so-called period of stagnation, the government became apprehensive of the spirit of freedom that invariably arose in schools that encouraged free thought, even if it was free thought in mathematics only. Alexei Sossinsky (1989), one of the main contributors to the magazine *Kvant* and a former teacher in Kolmogorov's specialized boarding school in Moscow, notes that during the 1970s "[the school] became more and more like a preparatory course for students from the provinces, with the social background of the students playing an increasingly greater role in their acceptance to the school, and their actual aptitude for science playing a decreasing role." Many schools were closed or became inaccessible, and the mathematics community exerted considerable effort to rescue others (see, for example, the report on the importance of mathematics schools written for the regional committee of the Communist Party by the well-known geometer V. A. Zalgaller, Perepiska, 1974, pp. 58-64).

In "ordinary" schools, work with gifted students was supposed to take place in elective courses. With a few exceptions, the specialized schools initially covered only the last two years of the students' education. Only considerably later did specialized schools expand to encompass the last four years. Meanwhile, in "ordinary" schools, work with students who had a special interest in mathematics was supposed to begin much earlier. Special handbooks were published for use in elective courses (for example, Nikol'skaya, 1991). These included chapters on such topics as computing in different number systems, construction problems, absolute value, graphs and functions, etc., and contained problems and theoretical material that was not found in school textbooks.

It is impossible to provide a detailed overview in this chapter of what has happened in Russia in the past twenty years. With the beginning of Gorbachev's perestroika, in the second half of the 1980s, the government began to publicize and promote the achievements of the mathematics schools: their success at preparing highly-qualified professionals was self-evident by this time, and former fears of the spirit of liberalism became irrelevant. The number of so-called schools with an advanced course of mathematics began to grow rapidly (for example, in St. Petersburg, in the beginning of the 1990s, approximately one tenth of the student body graduated from schools of this type, Karp, 1993). At the same time, the large-scale emigration of the technical-scientific elite (Grigorenko, 2000) severed the traditional and important ties between schools and their graduates. Moreover, the transformations that ordinary schools underwent during this time were not always beneficial to mathematics education.

The Soviet system of working with the mathematically gifted exerted great influence on many countries, first and foremost on countries of the former Eastern Bloc, which imitated the Soviet system in many respects (although the considerable influence exerted on the Soviet system by other mathematics education systems, for example the Hungarian one should also be noted (Vogeli, 1997)). Grigorenko (2000) has criticized this system for being "atheoretical." Indeed, it was not grounded in any psychological theory: it was a system created by mathematicians and mathematics educators. Psychologists, who had been the victims of an ideological crackdown in the 1930s, had far fewer opportunities to make themselves heard on the topic of educating the gifted than did subject matter experts. The psychological studies that appeared during this time, including highly significant works, such as those of Krutetskii (1976), were conducted largely in the shadow of what was taking place in the schools, where psychologists had little influence. Student selection usually took place either spontaneously in circle meetings (not all of the students who joined mathematics circles maintained their involvement with them—some lost interest, others found them too difficult); or based on the results of various exams and interviews, which took into account to some extent the students' psychological characteristics as well (B.M. Davidovich, personal communication, 2006), but which were carried out in any case by practicing mathematics educators.

Did the Russian system guarantee the "balanced development of the talented"? Did it necessarily develop the students' mathematical talents while providing a broader development of their complete personalities? Boltyanskii and Yaglom (1965) see the significance of the mathematics circles above all in the preparation of future mathematicians. The authors concede that mathematical talent can reveal itself even in the absence of such preparation, but they add that the constantly growing demand for qualified mathematicians cannot be satisfied spontaneously without special measures. Thus, the central pedagogical task is specifically the preparation of specialist mathematicians. This goal, however, was given a rather broad interpretation. A pamphlet describing the boarding school at Moscow State University (Kolmogorov et al., 1981), for example, devotes considerable attention to performances given at the school by singers and actors, to the school's classical music club, and to its literary societies. At the same time, it is difficult not to agree with Grigorenko (2000) when she notes that the humanities were frequently in an inferior position in the Soviet school system in general, despite (or rather, precisely because of) the government's incessant effort to augment the role of ideology in education, which manifested itself predominantly in the teaching of the humanities.

Another phenomenon developed gradually in mathematics schools and mathematics circles, which Fomin (1994) calls "mathematical athletic professionalism" (p. 10): an excessive emphasis on the "athletic" or competitive side of learning, that is, on the winning of awards and prizes in mathematics Olympiads. Consequently, class and circle meetings came to devote more and more time and attention to training students for the Olympiads and to solving typical Olympiad problems, to the detriment of the students' general development, even of their general mathematical development. As Fomin notes, "since these class and circle meetings are motivated by the desire to perform well in the Olympiads, children who cannot think on their feet or who do not like solving tricky problems can be left out of the general educational process and become disillusioned with mathematics in general" (p. 11).

DISCUSSION AND CONCLUSION

Russian education of the mathematically gifted may be criticized for the excessive role of subject matter and for the overwhelming influence exerted on it by professional mathematicians. Both these factors can make Russian mathematically gifted education look like a return (although, of course, only a partial one) to the model of the professional school. This type of school existed for centuries, serving specifically to prepare specialist mathematicians, armed from an early age with a panoply of techniques and devices and trained to solve professional problems, rather than exposed to mathematics in the context of a general development of their personality, which encompasses more than an aptitude for mathematics. Suppressing the other sides of the student's personality, or failing to take them into account, can have a lethal effect on the student's mathematical talent as well. Note, incidentally, that the professional applied mathematician of the 16th century automatically turned out to be a broad expert because specializations in our narrow contemporary sense simply did not exist at the time, while today the danger of a professional narrowness of scope is far greater.

By contrast, American education of the mathematically gifted may be criticized for the excessive influence exerted upon it by psychologists, or more precisely, for the insufficient involvement of mathematicians. This manifests itself in a paradoxical fashion in the fact that, in this system mathematics is often viewed as a sequence of courses that must be taken, rather than a particular way of thought that is characteristic of mathematicians, which they apply in dealing with any subject matter, and which is precisely the skill that mathematics education must seek to develop in students. People who have no connection with mathematics often see more advanced mathematical reasoning as inextricably linked to more advanced courses. This is why non-mathematicians sometimes argue that mathematics education must give students earlier exposure to fundamental college courses. Excessive concern for the general problem of development without sufficient attention to —or better yet love for — the specific subject matter can steer the students' development away from mathematics.

Naturally, we simplify the situation when speaking about the Russian or American tradition as a whole. A wide range of experience is available in both countries. What can be seen even from the most brief and schematic description of past history, however, is that the education of the mathematically gifted always balances between various alternative paths that directly oppose one another. We have described an opposition between a general approach and an approach more oriented around the mathematical subject matter. Other oppositions are no less significant: *individual vs. large-scale* teaching, *a local vs. national* system, *formal vs. informal* selection of gifted students, and so on.

The education of a gifted schoolchild must always be individualized, because each personality is unique. However, people begin discussing the education of the gifted seriously only when education becomes a large-scale phenomenon, and when the need for large numbers of mathematically gifted individuals arises. How can these individuals be identified? Not objectively, in other words, based on personal observations, however unreliable they may be at times, and however impossible it may be to compare individuals in this way on anything approaching a large scale. Or perhaps objectively, that is, based on some kind of tests, although it is not always possible to achieve consensus on what the tests are supposed to measure. Should there be general, national programs for working with the gifted? But these would acquire their own bureaucracies and meaningless routines. Should the education of the gifted be the business of the school, the district, a group of enthusiasts, with all their obvious limitations in experience and resources?

History does not and probably cannot provide a single answer to each of these questions. It can tell us only about the answers that have been given to them in the past, successfully or unsuccessfully, and about past experience. Acquaintance with this experience is useful, if only because it enables us to appreciate the complexity of the issues we confront. What may probably be learned from history, however, is that there has always been an inextricable link between the education of the mathematically gifted and the education of all students in general, even if this link has not always been recognized. When only few students are taught mathematics, fewer mathematically gifted students are identified. On the other hand, we can also see that the teaching of the mathematically gifted, past and present, has had a direct influence on the teaching of all students in general (witness the worldwide influence of Polya's ideas, which were clearly influenced by the Hungarian system of educating students interested in mathematics).

In this chapter we have only touched upon the history of the accumulation of methodological ideas and materials. Meanwhile, this history is important and useful both to the historian and to the practicing teacher. The experience of teaching through problems, of presenting the same topics in different ways, various approaches to telling students about contemporary applied problems, and finally simply collections of problems assembled at different times and in different countries form a vital legacy that must not be lost or forgotten. The experience of the ancient mathematician can live on in today's classrooms. This is what unifies many centuries of mathematical history. This history will continue, giving rise to new problems and new attempts to solve the old ones. The ability to stop and look backward is essential for moving forward successfully.

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NOTES

1 This and subsequent translations from Russian are by the author.

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ABBREVIATIONS

TSGA SPb –Tsentralnyi gosudarstvennyi arkhiv [Central Government Archive], St. Petersburg.

TSGA IPD SPb – Tsentralnyi gosudarstvennyi arkhiv istoriko-politicheskikh dokumentov [Central Government Archive of Historical-Political Documents], St. Petersburg.

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