Children Learn Mathematics
A Learning-Teaching Trajectory with Intermediate Attainment Targets for Calculation with Whole Numbers in Primary School

Marja van den Heuvel-Panhuizen (Ed.)
CHILDREN LEARN MATHEMATICS
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A LEARNING-TEACHING TRAJECTORY WITH INTERMEDIATE ATTAINMENT TARGETS FOR CALCULATION WITH WHOLE NUMBERS IN PRIMARY SCHOOL

developed by the TAL Team

Editor
Marja van den Heuvel-Panhuizen

FREUDENTHAL INSTITUTE (FI) UTRECHT UNIVERSITY
AND
NATIONAL INSTITUTE FOR CURRICULUM DEVELOPMENT
(SLO)

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Towards educational quality
Improving the quality of education is an important endeavor of educational policy and TAL aims to contribute to this. TAL is a project initiated by the Dutch Ministry of Education, Culture and Sciences, and carried out by the Freudenthal Institute (FI) of Utrecht University and the Dutch National Institute for Curriculum Development (SLO), in collaboration with the Rotterdam Center for Educational Services (CED).

The quality of education can be improved in many ways. TAL proposes to do this by providing insights into the broad outline of the learning-teaching process and its internal coherence. It aims to be a support for teachers alongside mathematics textbook series. Furthermore, TAL can provide extra support for teachers if it is incorporated into a circle of implementation.

The TAL project aims to describe the intermediate attainment targets of primary school mathematics. These objectives represent a further development of, and a supplement to, the previously established core goals for the end of primary school.

A defining feature of the intermediate attainment targets is that they are embedded in a learning-teaching trajectory. This is also the reason for calling the project TAL, which in Dutch stands for Tussendoelen Annex Leerlijnen; in English this means Intermediate Attainment Targets in Learning-Teaching Trajectories.

Eventually, learning-teaching trajectory descriptions will be developed for all domains of primary school mathematics.

The present book contains the learning-teaching trajectory for the domain of whole number calculation.

The book contains one trajectory for the lower grades (kindergarten 1 and 2, and grades 1 and 2) and one for the upper grades of primary school (grades 3, 4, 5 and 6). This means that the book covers the learning process in this domain for children ranging from 4 to 12 years of age.

Work of many
Since July 1997, when the Dutch Ministry of Education, Culture and Sciences commissioned the TAL project, a great deal of work has been put into developing this learning-teaching trajectory. Many people have contributed and their help was of great value in bringing the learning-teaching trajectory description to its present form.

We would like to mention the members of the Resonance Group\(^1\), in particular, and the teachers and all others who in some way worked on the consultations.\(^2\) Their input has given the book a broad foundation and their reactions were of great value in reaching the description of the learning-teaching trajectory we now have.

An extra word of thanks goes to the schools\(^3\) who cooperated in our research that was needed to develop the learning-teaching trajectory.

Finally, we have to thank the TAL Advisory Committee\(^4\) that ensured the fine-tuning of policy with the learning-teaching trajectory descriptions for the other subjects. This committee also opened up ways for implementing TAL by offering possibilities for developing modules for informing school teams about the learning-teaching trajectories and in-service training modules for primary school mathematics coordinators or teachers who would like to specialize as such. The mathematics coordinators will play a crucial role in implementing the learning-teaching trajectories in school practice.
TAL as a source of inspiration
There were many indications of a great need for an overview of the teaching of calculation with whole numbers from kindergarten to grade 6. This book is an answer to this need. It represents an attempt to bring structure and clarity to this domain as a core part of the mathematics curriculum in primary school.

Although it is impossible for a learning-teaching trajectory description to offer a solution to all educational problems, we hope that this book will stimulate educational practice and inspire teachers towards significant professional achievements.

TAL Team, Utrecht, July 2001
Introduction

Contents of the book

Whole number calculation in primary school
This book describes the intermediate attainment targets and mathematics learning-teaching trajectory for the domain of calculation with whole numbers in the primary school grades.

Part I covers the lower grades of primary school (K - grade 2). In these grades the domain covers counting and calculating in the number range up to one hundred, although numbers over one hundred may also be used. Large numbers have always fascinated young children and this fact is certainly one that can be put to use in teaching.

Part II covers the higher primary school grades (3-6), when the number domain is extended and more specific ways of calculating are dealt with. This means that a gradual distinction is made between mental arithmetic, column calculation and algorithms, estimation, and using a calculator. These calculation forms are also taken as the basis for describing the learning-teaching trajectory for the higher grades. Rather than just one trajectory, Part II thus covers several trajectories.

A learning-teaching trajectory with examples
The description of the learning-teaching trajectory is characterized by the extensive use of examples. These examples are not intended purely as illustrations but can be seen as real-life examples capturing the essence of a certain aspect of mathematics education, as well as providing practical insight into the relevant teaching methods.

Intermediate attainment targets with a double perspective
The examples of student behavior and outlines of teaching activities are intended to give a helicopter view of the learning experience that children encounter in primary school.

Meanwhile, the crucial steps that the students take are continuously summarized as intermediate attainment targets. These targets have been conceived as landmarks towards which the teaching can be orientated.

To offer teachers an even firmer footing, the teaching aspects are simultaneously illustrated with the formulation of the intermediate attainment targets, which summarize how the teaching can contribute to achieving these targets. This double perspective can be recognized from the two different boxes drawn up for formulating the intermediate target and for the teaching framework.

Overview of the chapters
After the introduction there is a chapter including background information about learning-teaching trajectories with intermediate attainment targets. This chapter is intended for those who would like to learn more about this new phenomenon in Dutch education. It includes a discussion of how learning-teaching trajectory descriptions can be put to work in the practice of teaching.

Part I
The first chapter of Part I covers the pre-school years. It contains a short outline of the developments that take place during this period in the learning domain that deals
Part II contains an introductory chapter which forms the link between the learning-teaching trajectory descriptions for the lower grades and those for the upper grades. The chapter summarizes the learning process in the lower grades and reveals a glimpse of how that process develops further in the upper grades.

However, the core of the learning-teaching trajectory in Part II is formed by the chapters on mental arithmetic, column calculation and algorithms, and estimation. The chapter on numbers and number relationships is more a description of the general number context in which the different forms of calculations are developed. Even though these different forms are described in separate chapters, they overlap to a large degree. The ultimate goal of learning to calculate with whole numbers is that students choose the best calculation way to solve a certain problem. The chapter on using a calculator should be seen in this light. A general chapter on the teaching framework closes the description of the learning-teaching trajectories and this is where the trajectories come together again. This chapter links the intermediate attainment targets of the trajectories for calculation with whole numbers and the core targets for mathematics as a primary school subject. This is put together with a general description of the teaching framework required to achieve these targets. And finally, part II of the book also contains a glossary.

The book concludes with an Appendix containing the official Dutch list of core goals for mathematics at the end of primary school and a list of notes.

TAL gives an overview

No single description of the learning-teaching process can do justice to the subtleties of what happens in practice. This is also true for the TAL learning-teaching trajectory. Nevertheless, this description of the learning-teaching trajectory has some inherent value.

Aim of TAL

The aim of TAL is to offer clarity and insight into the educational route followed by primary school students learning mathematics. Its reference point is the attainment targets of mathematics to be achieved at the end of primary school. In a sense, TAL sketches out the way towards them.
The description, however, is not intended as a straightjacket: real learning processes are too complex for that and they are never repeated in precisely the same way. This becomes clear as soon as one looks at the learning processes followed by different children. Nonetheless, if the learning process of a whole class is traced, certain broad outlines can be identified.

Covering a wide range
The book describes this communal learning-teaching trajectory encompassing the majority of students for the domain of whole numbers, and it does not attempt to accommodate very weak or very able students. Although the focus is on the main lines of the learning process, the individual differences in the learning routes are not ignored. As a result the learning-teaching trajectory has a certain “bandwidth.” This means that, within certain margins, there can be more than one way of reaching the same end. This is most clearly seen in the descriptions of the intermediate attainment targets, which are intended as markers along the trajectory and which are coupled to different levels of thought and action. The exact nature of these intermediate attainment targets is put into words and pictures with examples of the teaching of certain core activities. These illustrate, in particular, how varied students’ behavior can be. The extra footing this provides for the teacher lies in that, alongside descriptions of the various strategies children adopt, it shows which approaches have potential and which run the risk of leading to a dead end.

Limitations
Teaching-learning processes are not isolated processes within a single school subject or even within part of a subject. In this learning-teaching trajectory for whole number mathematics, the links to other subjects may not always be self-evident or may only be referred to in passing. The same applies to the links between this and other topic domains within the subject of school mathematics. Clarity requires the setting of certain boundaries and a learning-teaching trajectory describing links from everything to everything else, even if it were possible, would scarcely fit into a single book.

Longitudinal perspective
Undeniably, limiting one’s field of view to calculation with whole numbers has its disadvantages. And yet this is precisely what underscores the power of this approach—the power of a helicopter view, the possibility of grasping, in a few large-scale steps, the course of development that takes place in the primary school period and of seeing the links between the present and both what is to come and what has already passed. In fact, teaching without such a broad overview is not even possible. It must form part of the “educational mental map” of any teacher if they are not to get lost in details. It is only with this in the teacher’s mind that the full extent of the links that provide true quality in education can be achieved. It is hoped that the trajectory described in this book will contribute to forming such an overview.

Beyond textbooks series and beyond tests
Gaining this longitudinal perspective requires putting some distance between oneself and everyday teaching practice, but not so much that one loses touch with reality. This is why the practical examples in this learning-teaching trajectory description are given such prominence. They render the broad outline concrete and recognizable so that it can be used as a frame of reference for making didactical decisions. The learning-teaching trajectory description is, in fact, not a textbook in the usual sense but rather a source of inspiration for transcending the textbook. In the same way, the intermediate attainment targets can serve as guidelines for assessment. Since the targets are incorporated into a learning-teaching trajectory, they can be reached at different levels and at different times, and they should not therefore be used as a simple yardstick for measuring the achievements of individual students. Similarly, it is true that the intermediate attainment targets are intended more as a source of inspiration than as a strict way of monitoring the students’ progress.

A new educational phenomenon
Learning-teaching trajectory descriptions represent a new phenomenon in Dutch education—but on the other hand they have also a long history in the Dutch subject matter connected, didactical approach to mathematics education.
The descriptions serve as guidelines for teaching and, as such, reach further than the attainment targets set for the end of primary school. They not only show the aims for primary school, they also describe the process that leads to these core targets and the landmarks that can be recognized along the route.

A recurring pattern of interlocking transitions to a higher level forms the connecting element in the trajectory. The form chosen for the learning-teaching trajectory description is also new. It is not a simple list of skills and insights to be achieved, nor a strict formulation of behavioral parameters that can be tested directly. Instead, a narrative description is given of the continued development that takes place in the teaching-learning process.

Hopefully, it will also be a compelling story, one that can provide the teacher with mental support with real benefit in the classroom.

In short, the learning-teaching trajectory:

– provides an overview and thus a basis for putting teaching into practice but is not just a textbook series
– sets out important signposts but is not a pre-ordained learning route
– renders the differences between children recognizable but does not describe a learning route for individual students
– serves as a source of inspiration for didactical action but is not a didactical handbook
– can benefit education but is not the only way to improve the quality of teaching.

**TAL as a hold for teachers**

TAL is primarily intended to offer teachers a reference framework for taking didactical decisions. How much help a method provides in realizing teaching, teachers will still have to make their own decisions. This applies even more so if one wants to adapt teaching to the level of particular children and to create room for their own contribution. The teacher can regularly invite children to put forward their own ideas and solutions. These are then discussed in the class and the various solutions can be seen in relation to each other. This joint thinking process puts the proposed solutions at the command of the whole group and creates possibilities for booking common progress. The children can thus experience, in a natural way, how they can proceed based on what they already know and understand.

In such an educational set up, it is of utmost significance that the teacher can assess and clarify the children’s own ideas and solutions. The teacher must provide guidance in the exchange process and take various considerations into account with a view to the evident short cuts and raising the level of the children’s work.

This can only work well if the teacher has a good overview of the learning trajectories and knows when certain ideas and solutions can be expected. The longitudinal perspective then provides the teacher with the possibility of guiding the learning process. This means that the teacher can pave the way towards the short cuts and higher levels based on what the children propose.

It is in just these points that the descriptions of learning-teaching trajectories aim to give support. Firstly, by describing the relevant children as clearly as possible and thereby offering a handhold for recognizing and evaluating the children’s approaches and ideas. The approaches that may materialize in such learning-teaching situations are also described. And secondly, by offering didactical handholds with which the short cuts and higher level of the group can be achieved, while taking individual possibilities and differences into account. It is in this light that the intermediate attainment targets have a clear purpose: they are signposts for the learning-teaching process, for both the long-term perspective and for the micro-didactical aspects.
Learning-teaching trajectories with intermediate attainment targets

Descriptions of learning-teaching trajectories and the intermediate attainment targets that lie along them are new to Dutch education. This chapter explains what is meant by a learning-teaching trajectory in the context of the TAL project and how the intermediate attainment targets are regarded. Attention is also paid to the broader context of TAL—the so-called “implementation circle” of TAL—within which this book plays a central role. In looking at how the two are related, it becomes clear that learning-teaching trajectory descriptions could have a variety of applications in educational practice. Finally, this chapter will also deal with the ideas about mathematics education that lie behind this approach. When describing this general information, the chapter focuses on one particular learning-teaching trajectory—that of calculation with whole numbers—in detail. The examples given refer to components of this learning-teaching trajectory and there is an overview of this subject matter domain.

The learning-teaching trajectory description came into being from the constant re-thinking and re-writing of a series of draft versions supported by classroom investigations and consulting all kind of experts on mathematics education. The broad composition of the TAL team, with expertise in teaching mathematics to young children, teacher education, in-service training and counseling of teachers, education development, information technology, and the development of assessment tools formed the background to the various revisions.

In the course of putting together this learning-teaching trajectory description, the team developed a clearer view of learning-teaching trajectories, the overlapping levels that can be distinguished in them, and the intermediate attainment targets that can be reached.

What is meant by a learning-teaching trajectory?

The term learning-teaching trajectory
Put briefly, the learning-teaching trajectory describes the learning process the children follow. It should not be concluded from this, however, that it only contains the learning perspective. In this book a learning-teaching trajectory has three interwoven meanings:

– a learning trajectory that gives a general overview of the learning process of the students
– a teaching trajectory, consisting of didactical indications that describe how the teaching can most effectively link up with and stimulate the learning process
– a subject matter outline, indicating which of the core elements of the mathematics curriculum should be taught.

At the same time, a learning-teaching trajectory should not be seen as a strictly linear, step-by-step regime in which each step is necessarily and inexorably followed by the next. It should be seen as being broader than a single track, because it needs to do justice to:

– the learning processes of individual students
– discontinuities in the learning processes; students sometimes progress by leaps and bounds and at other times can appear to relapse
– the fact that multiple skills can be learned.
simultaneously and that different concepts can be in
development at the same time, both within and outside
the subject
– differences that can appear in the learning process at
school, as a result of differences in learning situations
outside school
– the different levels at which children master certain
skills.
In short, there is sufficient reason to talk about a learning-
teaching trajectory having a certain bandwidth.

A learning-teaching trajectory description in levels

Levels as a basis for cohesion
The learning-teaching trajectory description describes not
only where the children should end up but also the broad
path they will follow in getting there. This path does not
consist of isolated knowledge elements but rather consists
of skills and insights between which cohesion gradually
builds up. What is learned in one phase, is understood and
performed at a higher level in a later phase. To make
proper use of the TAL learning-teaching trajectory
descriptions, it is essential to be clear about the stratified
nature of the learning process. Properly understood, the
different levels can supply a large part of the cohesion in
the learning-teaching trajectory.

Levels as a basis of a communal trajectory
The fact that the students are able to understand the same
subject matter at different levels forms the basis for the
communal learning-teaching trajectory. This means that
children can work in one and the same subject matter
domain while the are at different stages of their
development.

Levels as a basis for didactical holds
It should also be noted that a learning-teaching trajectory
description based on levels provides teachers with holds
that allow them to monitor the development process of the
students. This, in turn, allows them to adapt the teaching
towards level raising.

Sequence of levels
One new element in the levels terminology is that it goes
further than the conventional schema of moving from the
material-based to the mental manipulation with numbers
and in some places there is a radical break with this
sequence. The idea that learning processes always follow
a fixed path from the material to the mental can easily lead
teachers astray. To take just one example, children do not
necessarily have to learn resultative counting by pointing
at building blocks first before they can enter the mental
level of resultative counting through verbalizing their
actions. This process can in fact work in reverse—from
the mental to the material. The children will, after all,
already have fragments of the counting sequence
memorized as a rhyme, as well as preliminary ideas about
the properties of certain quantities, when they set about
finding the number of blocks by counting or by some
structured procedures.

Domain-related levels
In TAL, the levels are not approached as universal levels
of thinking. Instead, the specific forms that can emerge in
the different domains of mathematics are taken into
account, for example, in the domain of working with
numbers up to twenty, a different set of levels can be seen
in counting than in calculation, as shown in the following
diagram.

<table>
<thead>
<tr>
<th>Context-bound counting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Object-bound counting</td>
</tr>
<tr>
<td>Calculation by counting</td>
</tr>
<tr>
<td>Calculation by structuring</td>
</tr>
<tr>
<td>Formal calculation</td>
</tr>
</tbody>
</table>

An important practical advantage of domain-related
levels is that they provide teachers with firmer handholds
for taking didactical decisions than those given by an
overly abstract concept of general levels.
Specific levels in development

With regard to the students, the levels do not represent any kind of general stage in their development and can only be considered dynamically. The children can function at different levels in different content domains or in parts of them. It is important, however, to realize that the levels terminology does not contain an exact description of what the children can do but is more a way of monitoring their learning process.

Teaching changes during the trajectory

As well as showing the way to a certain attainment target, the description of the learning-teaching trajectory makes it clear that the learning process—and therefore also the teaching—does not proceed in the same way in each phase of development. For example, take the fact that kindergarten children benefit most from a stimulating environment containing integrated education moments. This contrasts with the upper years of the primary school where there is more course-based teaching in which mathematics is taught as a more or less separate subject, while not losing sight of its connections with other subjects.

A learning-teaching trajectory cannot exist in isolation

Getting a grip on a complex process nearly always requires putting aside its complexity. This also applies to describing learning-teaching trajectories. Without setting certain limitations, it is not easy to see how the different trajectories link up. At the same time, focusing on one particular domain should not mean losing sight of how it connects with other areas of mathematics.

The learning-teaching trajectory for whole number calculation cannot be seen in isolation from the learning-teaching trajectories for measurement and geometry, and this is why the learning-teaching trajectory sometimes provides holds for connection points. For example, the teaching of odd and even numbers might include placing a mirror next to a number of blocks, which brings in geometry. Because it is so important not to overlook links with other subjects, language and mathematics will be tightly interwoven, particularly in the early primary-school years.

In light of this, an even more fundamental question is to what extent the subject component can be disconnected from, firstly, the child’s cognitive development and, secondly, its social and emotional development. This has a simple answer: all these lines of development are inseparable but distinguishable. It is the act of looking at them separately that provides a broader view of a subject and that allows teachers to see how specific lines interrelate and can be integrated. One could say that if integration is successful then birthdays would not be celebrated in class simply to teach the children to count their ages in years.

Longitudinal sections and cross-sections

The learning-teaching trajectory description not only provides teachers with an overview of the educational route, it can also provide a better view of the state of affairs in the class. This cross-section of the students’ abilities and insights can be better understood from the perspective of the longitudinal section and vice versa.

How should the intermediate attainment targets be interpreted?

Intermediate attainment targets as benchmarks

The intermediate attainment targets used as benchmarks in the learning-teaching trajectory create a series of reference points against which the development of the students can be assessed. They act, in other words, as calibration points.

The levels terminology used in the intermediate attainment targets shows not only how far the students have come but also offers some handholds for how the teaching might proceed. In addition to this guidance for assessment, the intermediate attainment targets simultaneously provide landmarks for the orientation of
the teaching. Formulating these objectives also throws light on the teaching methods at each stage of the learning process.

**No disconnection of trajectories and targets**

With the need for an overview in mind, the benchmarks incorporated into the learning-teaching trajectory are formulated explicitly as intermediate attainment targets but they should not be used in isolation as a kind of checklist. Deterioration of the teaching would then become unavoidable. The summary of the targets cannot, of course, fully reflect the rich repertoire of students’ behavior at a certain stage of development. The same is true for the description of the teaching connected to the intermediate attainment targets. Both can place emphasis and state priorities but, by themselves, have nothing like the power needed to set the intended learning process in motion.

**Coherence between intermediate targets and general goals**

The intermediate attainment targets formulated in the learning-teaching trajectory need to be seen in the context of the general goals for primary school mathematics. The latter, in turn, form one component of the general educational goals. To underline this cohesion, general teaching frameworks have been formulated at the end of each phase of the learning-teaching trajectory.

**Relationship with the final core goals**

The intermediate attainment targets incorporated into the learning-teaching trajectory for whole number calculation represent a further development of, as well as a supplement to, the core goals for mathematics set by the government for the end of primary school (see Appendix). This involves, in particular, the five general goals formulated for mathematics and the specific core goals that form part of the domain of calculation skills. The first seven of these describe the domain of calculation with whole numbers in the lower grades of primary school.

Putting aside the legal status of the final core goals, both they and the intermediate attainment targets show what should be both desirable and achievable for virtually all students. Just as with the final core goals, it should be remembered that the list of intermediate attainment targets gives only a general indication and can, within certain margins, be filled in on an individual basis. The size of this margin has a lot to do with the difficulty level of the domain at issue. For instance, in the case of decimals and fractions, which not all the students will be able to deal with, there will be a greater margin than for elementary (mental) calculation up to one hundred, which belongs to the core subject matter for all children at primary school.

Another area where the final core goals and intermediate attainment targets correspond is that particular intermediate attainment targets are sometimes the same as the final core goals. Examples of this are the counting sequence, and multiplication and division tables. One could say that in these domains the students reach the final core goals before the end of primary school.

**Intermediate attainment targets with flexibility**

If the description of aims in the final core goals can be said to involve some flexibility, this is even more true for the intermediate attainment targets. The nature of the levels means the intermediate attainment targets can be reached in a variety of ways.

By the end of grade 2, for example, the students should all be able to solve a problem such as $83 - 39$, but they can differ significantly in how they go about this:

- some children do the calculations in their heads
- others can only solve the problem by making a sketch of the number line.

The strategies used for purely mental arithmetic can also vary and so can the strategies for working it out on the number line:

- some children will count backward in small steps ($83\ldots, 82, 81, 80$, take away another $6, 74\ldots, 64, 54, 44$)
- others use a more structured approach and count backward in large steps ($83 - 30, 53, 53 - 9, 44$)
- yet others use a clever compensation strategy (first take away $40$, then add $1$).
If the same problem is presented as a context problem, the list of possible strategies is even larger because the children can then also make use of context-connected strategies. In principle, all the combinations of the ways of processing and the calculation strategies given above are possible, but it is obvious that certain ways and strategies will be used together more often. Children who still calculate by counting will mostly do so with the help of counting materials.

The various ways shown here for solving a problem like $83 - 39$ are by no means complete, but they do show that an attainment target cannot be established simply by agreeing which problems students should be able to solve at any given point. In this respect, the levels at which problems are solved are more important than the nature of the problems themselves. It should be clear that in the TAL learning-teaching trajectory the various levels are conceived more as conceptual levels—levels in understanding—rather than as achievement levels. The latter would involve a level terminology indicating what likelihood there is that the students are able to complete a given number of problems of a certain type without error. How the students actually solved the problems would not be considered.

**Relation to school years or grades**
The learning-teaching trajectory description gives only an approximate indication of the time course. The trajectory and its associated intermediate attainment targets are described for two successive school years. There is one chapter for years 1 and 2 of kindergarten, one for grade 1 (and 2) and one for grade 2 (and 3). Although the last two chapters concentrate on teaching in a particular year, they continue into the curriculum of the following year. With regard to the students, this double grade indication shows that although the intermediate attainment targets generally apply to one year, there will be some students who only achieve them in the following year.

In phasing the intermediate attainment targets an excessive precision was avoided, since it was felt that such a precision could easily lead to putting unwarranted pressure on teachers to carry out continuous testing to see whether their students were keeping up to schedule. Quite apart from the fact that this would be neither feasible nor desirable, such a strict timetable of objectives would not correspond with the natural course of learning processes. For this reason, only general indications are given. These provide both teachers and students with some space to reach the intermediate attainment targets, within a certain range, in their own time schedule. The intermediate attainment targets are therefore flexible in both length and breadth. This is the bandwidth referred to earlier.

**The breadth of the learning-teaching trajectories**
The learning-teaching trajectory described for whole number calculation is designed for the great majority of students attending the lower grades of primary school. In other words, it has been made broad enough to accommodate all the students. The goals can be adjusted in both timing and level for individual students. Even more important than this, however, are the adjustments made to the related teaching. The differentiation in the learning-teaching trajectory for this basic subject domain of whole number calculation is sought primarily in the teaching side rather than by adjusting the intermediate attainment targets.

**Possibilities for using learning-teaching trajectories in school practice**
A learning-teaching trajectory description is not a handbook for everyday practice but it can offer support in realizing good mathematics teaching. This can be brought about by, among other things, opening up an across-the-school discussion about mathematics education. With the help of a learning-teaching trajectory, the boundaries of the grade classes can be broken down and a language is provided for discussing mathematics with the whole teaching team.
At the same time, the cohesion between the work of the different grades will become more evident. The extent to which schools may go along with this partly depends how far they have a collegiate structure with room for professional didactical discussions. The head of the school or a mathematics coordinator who can guide and stimulate this process is indispensable.

What teachers actually do with the descriptions of the learning-teaching trajectories in terms of actual teaching will vary from school to school and between individual teachers. This book, however, is not the place to discuss this in detail or to establish fixed rules of use: the idea of working with learning-teaching trajectories is something that first has to take shape in practice in the years to come. In anticipation of this development, however, several possibilities for how learning-teaching trajectories can be used in practice can be considered.

Discussion about didactically difficult areas
Because the learning-teaching trajectories can clarify how certain skills and insights work together—and particularly in this case because of the way the conceptual levels are set out—didactical trouble spots become less exclusively the terrain of a single grade. Developing a feeling for numbers, learning number relationships and gaining insight into the sizes of numbers are potential stumbling blocks for children in all grades and teachers of different grades can learn much from each other in this respect. The same could be said for practicing the basic skills, something that also appears in all grades. In holding team discussions about these trouble spots, the learning-teaching trajectories can serve as a frame of reference.

Holding student reviews
The learning-teaching trajectories and the intermediate attainment targets also offer a framework for structuring student reviews. The advantage is that the reviews do not have to be confined to the sorts of problems the students can or cannot do but attention can instead be given to the strategies being used. In the learning-teaching trajectory strategies are described extensively.

Making didactical decisions and developing materials
Although the level of mathematics textbook series available in the Netherlands is adequate, they remain nothing more than books until a teacher turns them into education. First of all, a teacher will make a selection from the activities and problems offered in the textbooks. But this means, more often than not, that they end up developing their own new materials for practicing. In general, the oral practice falls almost entirely under the responsibility of the teacher and cannot be found in the textbook. The learning-teaching trajectories can help here both as a frame of reference and as a source of ideas. In other words, the learning-teaching trajectories can help the teacher to rise beyond the limitations of the textbooks.

Organizing a parents’ evening
Keeping the parents informed means being able to show examples that reveal something of the general working methods in the classroom. Leafing through a textbook is not always the best way of doing this. It might evoke recognition of the work the children bring home with them, but often it does not make clear what is being taught or how. Showing part of the learning-teaching trajectory illustrated with video clips from classroom practices.

Choosing textbook or student monitoring system
Choosing a new textbook series or making a decision about a monitoring system always involves reflecting on the goals of the teaching and the approach it uses. What do we really want to achieve with the teaching? And what type of teaching will best serve our students and our own goals?
Our own experiences form the most important criteria here, but they first need to be made explicit with the help of a particular frame of reference, such as that provided by the learning-teaching trajectory description. This frame shows what we need to look for when purchasing a new textbook series. How the learning-teaching trajectory descriptions will work in practice for each teacher will depend on the nature of the school and classroom practice in question.
TAL’s implementation circle

Educational development in mathematics in the Netherlands has not been an isolated enterprise. Since the 1970s, it has developed a tradition and its own strongly integrated approach. The development of new didactical insights has always gone hand-in-hand with forging links to teacher education, in-service training, counseling of teachers, the development of textbook series and, last but not least, the practice of teaching itself. Primary school mathematics education in the Netherlands has a good infrastructure, which makes it possible for these echelons to work effectively and to intercommunicate.

The work of TAL does not end with descriptions of learning-teaching trajectories. If these are to help in improving mathematics education, connections have to be made to schools and classroom practice. The learning-teaching trajectories need to become part of teachers’ thinking. Only then can the trajectories act as a stimulus to the teachers’ work.

All of this cannot be realized with one book only. In-service training and counseling are necessary, and links must be made from TAL to the initial teacher education courses because here is the foundation laid for insight into the educational routes. Furthermore, connections also need to be made with textbook series development, teachers’ choice of a new textbook series, and with the development of assessment tools and teacher-conducted assessments.

For this reason, together with the development of learning-teaching trajectories the TAL team is also working on implementation materials to be used for:
- teacher education
- in-service training and counseling
- education development related to monitoring students’ progress and evaluating materials and textbook series.

The learning-teaching trajectory description should be used as a starting point for putting together the implementation materials, which can be seen as a circle surrounding the learning-teaching trajectory descriptions.
Realistic mathematics education as a starting point

The educational process described in the learning-teaching trajectory reflects the so-called “realistic” approach to mathematics education. All the aspects of the domain-specific education theory underlying this approach—which has been in development since the 1970s and which has strongly influenced almost all the textbook series used in Dutch primary education—can be found in the learning-teaching trajectory descriptions. This applies just as much to the view of the subject as a “meaningful human activity” and the goals of mathematics teaching, as to ideas about learning and teaching of school mathematics.

Realistic mathematics education is by no means a closed book and it continues to be developed. Every new experience in class or in teaching an individual child can lead to a sharpening, adjustment, or shift of emphasis within the teaching practice. It should not be surprising therefore that this particular learning-teaching trajectory description, written as it was by a broad team with contributions from teaching practice, contains some new elements of realistic mathematics education. One example of this is the idea of domain-related levels of understanding.

The goals

Apart from developing a critical mind and learning to trust their own thinking power, the students will also need to be helped to discover joy in mathematics.

How the goals can be achieved

If the students are to reach these goals, mathematics teaching will need to supply the tools they need and the lessons need to be directed towards the same goals. A rich learning environment within which children can develop mathematical skills and insights is a necessary but not sufficient condition for achieving these goals. A student’s active learning can achieve little without the contribution of the teacher to prompt and guide it. Realistic mathematics education can also be seen as the interaction of tutorial guidance and the student’s own constructs taking place during the student’s discovery or re-invention of mathematical knowledge and insights. All of which takes place under the direction of the teacher.

The starting point for this learning process is formed by context situations accessible to the children. It is in these specific environmental situations that children develop context-related strategies and notations that later become more generalized, particularly those based on contexts with a definite model character. This process of level raising through the use of models is helped along by interaction with other students. Confrontation with other peoples’ thinking and alternative solutions can push the student’s own thinking into new and fruitful directions. Besides providing opportunities for individual practice and assimilation, interactive group-orientated forms of teaching also lie at the heart of realistic didactics: they lead students to reflect on their own thinking and calculation methods and, in so doing, prepare the ground for level raising.
Overview of the subject domain of whole number calculation

This book lays out the learning-teaching trajectory for the subject domain relating to whole number calculation. This covers counting and calculation in the number range up to a hundred. As shown in the TAL symbol, numeracy plays a leading role in this.

Numeracy as an overall goal

The central place taken by numeracy in the learning-teaching trajectory for whole number calculation means that all the intermediate attainment targets in this trajectory need to be seen in the light of this goal.

In order to attain this goal of numeracy, children need to feel at home in the world of numbers and operations. They also need to be able to deal with different forms of calculation—mental arithmetic, estimation, and column calculation and algorithms, and if suitable the use of the calculator—and to be able to select the right method for the problem at hand, including the decision on whether or not to use a calculator. Consequently, numeracy is directed as much at solving numerical problems in everyday life as at the ability to find relationships and smart short-cut strategies within the arithmetic system. In this particular learning-teaching trajectory, all these considerations are shaped towards dealing with the field of whole number calculation.

The primary school learning-teaching trajectory

Translated to the different years of primary education, the following areas can be defined within the learning-teaching trajectory for calculation with whole numbers:

- Years 1 / 2 of kindergarten
  Counting and elementary number sense.

- Grade 1 / 2
  A feeling for numbers and calculation in the number range up to one hundred, with an occasional outing to higher numbers if there is a reason to do so. Command of addition and subtraction up to twenty and a start made with multiplication and division tables.

- Grade 3 / 4
  Calculation in the number range up to ten thousand. Mental arithmetic, estimation and algorithms. Command of multiplication and division tables.
Grade 5 / 6
Ability to assign meaning to very large numbers.
Gaining insight into numerical patterns and rules.
Divisibility properties, special numbers such as primes, triangular and square numbers, and elementary number theory.

What precedes primary school
Although the learning-teaching trajectory description actually starts in kindergarten years 1 and 2, this is obviously not where children really begin their mathematical development. By the time they enter kindergarten (age 4), they will have already accumulated a wide range of numerical learning experiences and it is this knowledge that forms the basis for what comes later.

Only when these early learning experiences are also taken into account can the right perspective and frame of reference be used to describe the learning-teaching trajectory in the kindergarten period and to decide on the most important intermediate attainment targets for this age group. This is why the learning-teaching trajectory description is preceded by a sketch of children’s development in dealing with numbers and quantities in the pre-school period.

What follows after primary school
When the learning-teaching trajectory for calculation with whole numbers is extended into secondary education, the topics will include working with negative numbers, large numbers such as powers of ten, infinity…
Part I – Lower Grades Primary School
“1953, the twelfth of October, Monday morning. Another enchanted day. Took the 9.43 bus to Loenen. Sunny but somewhat hazy. Autumn, Summer ennobled. Totally withdrawn into another world. At every twist and turn in the road, I thought, ‘What will come next, I wonder?’ (I must have cycled around these bends or passed them on the bus at least 200 times, perhaps 400 times, but these were completely new bends and all the willows and all the other trees were new.) (...) The road to everywhere. The Grutterstraatje and Brugstraatje roads, a single glitter. Bought a 22 cent cigar again at the corner. She recognized me at once and knew why I had come. Except that she didn’t have that pinafore from 25 years ago anymore, she said. The narrow bridge, the house built in 1502 (still lived in), the little street with the nursery school.”

(Nescio, Nature Diary 1946-1955)
Pre-school years
Emergent numeracy

Numbers and numeric phenomena provoke children’s interest from a very early age. Numbers represent a broad and intriguing area of knowledge that again and again offers new opportunities for invention. In this respect, the child’s own everyday life provides the best place for the learning process to be encouraged by meaningful interactions. All of this takes place before organized formal education even begins. Learning is, after all, a process that occurs primarily “from inside” and is driven by the child’s own natural curiosity, its urge to find out how things fit together, and through its need to imitate what grown-ups are doing.

By analogy with the term “emergent literacy” in language education, it is possible to speak of “emergent numeracy” as a similar process in which children gradually, and for the most part spontaneously, come to grasp the different meanings and uses of numbers and the knowledge of how numbers are interconnected. This emergent numeracy is characterized by the development of more or less autonomous elements of knowledge and the natural and spontaneous way in which these begin to cohere and form an interconnected whole. At this stage, the following elements of number sense can be distinguished:

- recognizing “two-ness”, “three-ness”, and “many-ness” as a property of a group of objects
- learning to recall the number sequence
- imitating resultative counting
- symbolizing by using fingers.

Not every child shows an equal interest in all things related to numbers and one child’s environment will differ radically from that of another. Consequently, some elements will develop more strongly in some children than in others. Nevertheless, they will all gradually come to see the relationships between these and other elements. Every child seems to find its own way to make sense of the fact that the three in “the number three in the row” has something to do with the three in “someone is getting three sweets” and the three in “someone is three years old.” This will be discussed further in the following section.

A helicopter view of the learning process

**Learning to distinguish two-ness, three-ness and many-ness**

The ability to recognize two-ness, three-ness and many-ness develops at a very early stage. By the age of about two, children generally give some indication that they know the difference between two and three in a collection of objects. This is most likely to happen in situations the child finds remarkable or surprising.

Adwoa (1;9) has reached the stage where she often shows that she knows that there are two of something. Mostly this involves objects that are directly available such as plates on the table, toy cars on the ground or people in the room. Today, however, she also shows that such knowledge also relates to things that have a less concrete presence. She is upstairs in the hobby room and sees the television in there.
“Two TVs, two TVs” she says.
There is, in fact, only one television in the hobby room but
Adwoa knows that since there is a television downstairs,
there really are two.

Another example.

Mara (2;2) has joined in with the other children from the
village to look for Easter eggs. It is an annual event in
which some fifty or sixty eggs are hidden in a field. Mara
finds the search difficult. She has little idea of the purpose
of all this activity or of the sorts of places where the eggs
might be hidden. With a bit of help, however, she finds two
eggs and walks around for a while with them in her basket,
repeatedly proclaiming:
“Two eggs.”
By the time she finds a third egg, the search is almost over.
This egg is also carefully placed in her basket, and when
she sees the three eggs lying there neatly side-by-side, she
immediately shouts out:
“Three! Three eggs...!”

At this point, resultative counting in the sense of
deliberately counting one-by-one to determine a quantity
means little to either Adwoa or Mara or, at least, they have
never given any sign that it does. Despite this, they are
quite able to distinguish two or three of something. This
often seems to be connected with being able to discern the
spatial structure in which objects are laid out.

Jorie (3;5) always gets five calcium tablets placed next to
her plate at lunchtime. She usually checks to see whether
there really are five but today there are only four. She
checks the tablets as usual. Although she already knows the
counting sequence to some extent (she can count up to
eight), she does not appear to use this knowledge. Instead,
she arranges the tablets into a rough square.
She then says:
“Hey, that’s not five. There should be another one!”
When she is given another tablet, she centers it above the
square of four and says
“Yes, now it’s five...”

The counting sequence as a verse to recite
Reciting the counting sequence is an activity that, in the
first instance, develops quite separately from the counting
of quantities. It is as if it were a special sort of verse to be
recited, learned and used in the context of games and
contests. “One, two, three!” and the child jumps off the
step. “One, two, three, four, five, six, seven, eight, nine,
ten. Ready or not, here I come!” But children also repeat,
extend and modify this verse on their own, without any
apparent connection to anything within the child’s sphere
of interest.

Sander (2;10) can be preoccupied with counting for up to
ten minutes at a time. For him, it appears to be a game in
itself and one he is already quite good at. At the same time,
the route he takes is far from straight and narrow. At the
moment, he mostly counts as follows:
“One, two, three, four, six, seven, eight, nine, ten, eleven,
Pre-school years

twelve, thirteen, fourteen, fifteen, eight, nine, ten, eleven, twelve...”
He consistently misses out the five. After the missing five, things go well for a while but after fifteen he switches back to eight and continues as before. As with many children, Sander constructs his own counting sequence and the imperfections in it are of little immediate importance to him.

Imitating resultative counting
Counting a quantity one-by-one to determine the total number (resultative counting) is a complex skill developed by trial and error. This is another case where children at an early age—often in the third year of life—can be fully preoccupied imitating what they see adults and older children doing without fully understanding the meaning of these activities.

Eva (1;9) is eating a sandwich, which has been cut up into small pieces and is lying on a plate in front of her. Before she starts eating, she counts the pieces but uses the numbers in the wrong order,
“One, five, three, four...”,
and this despite the fact that she is able to recite the counting sequence without difficulty. Then she starts to eat. The pieces are eaten in the order in which they were counted:
“This is one..., that is five..., that is three...”

Mara (2;2) often sees her older sister counting marbles. This fascinates her, and little by little she begins to count marbles herself. How exactly this is done and what the meaning of it all is, however, largely eludes her. Today she is drawing pictures with felt-tipped pens. While drawing, she removes the pens she needs from the pen box. When she has finished her drawing, she looks briefly at the unused pens still in the box. She starts to count them spontaneously, her finger moving criss-cross between the pens as she counts:
“One, two, four, six...; seven, eight, nine, ten. Ready or not, here I come!”

It seems that she has some vague intuition that counting is relevant in such a situation and in her own way imitates what she has seen her sister doing. Quite by chance, she switches effortlessly from one form of counting to another. The quasi-resultative counting shifts naturally and easily into something else she has seen her sister doing, namely playing hide-and-seek.

In situations like these, children still do not have a clear idea of what counting is really all about. Nor are they very good at distinguishing the various functions of counting. This will remain the case for some time, but eventually some understanding of the meaning and purpose of counting will emerge. An example is the resultative aspect of counting, that is, the insight that the last-mentioned number indicates the total of the quantity counted. Flowing naturally from this is the idea that there is something not quite right if counting in one way produces a different result from counting in another way.

Mara (3;6) sits on her mother’s lap. They are playing “Happy Families” – a card game played with hands of four cards. In front of Mara are four matching cards. She doesn’t have much idea of the game as such but plays along enthusiastically and in her own way. Eventually, she finds that she has in her hand the four cards that were originally on the table.
“Hey, four!” is her reaction as she spreads the four cards on the table in front of her.
Her mother says: “Yes, that’s right. There are four.”
Now, off the cuff, she starts counting the cards one by one in her accustomed way:
“One, two, three, four” (her finger pointing to the second card), followed by “six, seven” (now pointing to the fourth card) and “seven!” (looking at her mother) “no, four!”

Obviously the fact that she gets a different result each time creates a conflict for her but, and for the time being at least, she seems to have more confidence in her ability to recognize four objects at a glance than in the result of counting.

Symbolizing by using fingers
Apart from manipulating with and thinking about objects that are physically present, many children’s activities also relate to “imaginary” quantities. It is precisely in this kind of situation that children feel the need to represent quantities symbolically. Perhaps the best known example is a child using its fingers to show its age—sticking up three fingers to show that it is three years old—but fingers can also be used to show other quantities.

Eva (2;8) has a box of raisins in front of her.
Eva: “I want to have this many raisins” (she puts three fingers up).
Her mother: “How many is that?”
Eva: (she counts her fingers) “One, two, three!”
Her mother: “How many raisins do you want?”
Eva: “Two...” (she puts two fingers up, adds a third and says) “…three.”
Then she counts the five fingers on her other hand, points to the middle finger and says: “That’s three.”

Mara (3;4) usually gets told bedtime stories. Mostly two, but sometimes three and occasionally four. But tonight, I only tell her two stories to start with.

When they are over, she says:
“Two (she puts two fingers up) isn’t many, is it?”
After she presses the issue a bit, I read another story. When the third story is over, she puts up three fingers and says that three isn’t all that many either. After the fourth story, I say:
“Now we’ve had a lot, haven’t we? Four stories!” (and I put up four fingers).
She is not so easily beaten and says:
“But it could also be this many.”
And she puts up a full hand of five fingers.

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And she puts up a full hand of five fingers.
Changing amounts: using the counting sequence
Of course, using fingers is not the only way of varying an amount. As resultative counting steadily comes within a child’s conceptual grasp, the use of the counting sequence becomes an increasingly viable alternative. It is then used in a more formal manner, distinct from actual resultative counting.

Jorie (4;6): “Look, I’ve got lots of hats, haven’t I?”
Jorie is folding paper hats. Folding, cutting and gluing are the things that she likes doing best at the moment.
“I’ve got… one, two, three, four, five, six” (counting the hats correctly, one by one) “and I’m going to make even more.”

She then puts the hats on the piano in the other room. Back in the kitchen, she makes two more, and announces (while the first six hats are out of sight in the other room):
“Now I’ve got eight already…”
I ask: “How do you know that?”
She answers: “Well…, seven comes after six, doesn’t it, and after seven comes eight.”

However, the development of such a formal use of the counting sequence does not proceed at an equal rate in all children and not all children have progressed that far when they start in kindergarten.

Summary and preview
This emergent numeracy is in no sense a phenomenon that remains dormant until a child starts school. On the contrary, demonstrations of rudimentary numeracy can be seen even at a very young age. In this pre-school period, a child’s mathematical knowledge begins to develop in an individual way that depends largely on the environment in which it grows up. Initially, there is little connection between the various constituent knowledge elements. At a very early age—as early as three years in fact—a child already has an inkling of how to use a word such as “three” to show that there are three of something. Even so, the fact that this “three” has something to do with the three in “being third in a row” and the three in “three years old” will remain beyond the child’s intellectual horizon for quite some time.

Gradually, however, pre-school knowledge broadens. This goes hand in hand with a growing understanding of the correspondence between the different knowledge elements.

The “three” in “three Easter eggs” is increasingly perceived to be associated with the three used when counting markers one-by-one, with the three of “number three in the row” and, in due course, with the three of “three and three makes six.”

As these meanings and ways of using numerals become more interrelated, numerals also begin to acquire more meaning in their own right, as “things” one can think about and juggle with, irrespective of all the situations in which they occur in everyday life. Bit by bit, more formal arithmetic comes within reach. But usually a child is not that far when it starts in kindergarten. At that point, the process of expansion and interlinking of knowledge is still in full swing.

Knowledge can vary considerably from child to child. Some children are quite familiar with counting while others are not at all. The extent to which they are able to distinguish between the various meanings of numerals varies greatly, too.

The situations in which confusion emerges are well known.

Anita (4;3) is in a pancake restaurant with her father. They have just chosen a pancake from the menu.
“I want pancake twelve,” says her father to the waitress.
“And pancake seven for this young lady.”
Anita cries: “But I can’t eat that many pancakes…!”
The understanding of resultative counting that children have when they enter kindergarten differs enormously, too. Nevertheless, nearly all are able to say whether there are two or three of something in particular situations, though for many this will be the limit of their knowledge. For these children, resultative counting is still little more than an imitation game. But other children have come further. They realize, for example, that the last-mentioned number in one-by-one counting also indicates the total quantity, and some have even got to the point where they know the rules that have to be observed during counting, such as the fact that pointing to objects should stay in step with saying the corresponding numbers. They can already count to six or seven correctly.

This somewhat variable starting situation when children in all their diversity enter the nursery class might be described as the ground level. One of the most important educational tasks at kindergarten level is to create a rich learning environment, covering a broad socio-emotional, motorial and cognitive terrain, in which children can develop further. Numbers can play a challenging and meaningful role in these activities. As will become clear in the following chapter, from time to time numbers should be the focus of teaching. As a result, the child will be stimulated and encouraged to make sense of the numerical aspects of the world it lives in. Resultative counting that lays the basis for later numerical skills will increasingly come within the children’s grasp.
Kindergarten 1 and 2
Growing number sense

Children encounter numerous situations that bring them into contact with sounds, symbols and meanings that relate to numbers. Examples of these include counting rhymes, their ages, clocks and calendars, numbers on houses, buses and shirts but, most importantly, the counting and recognizing of small numbers of objects such as snacks and sweets or the number of dots on a die. From all this, they learn that numbers can have several different meanings and functions:

- “magnitude”, the quantity of five sweets
- “order number”, the fifth item in a row
- “measure number”, the age of five years
- “label number”, bus number five
- “calculation number”, two plus three is five.

Early on, these various aspects of number are relatively autonomous, but children do gradually develop an elementary number sense. They recognize different functions of numbers in everyday life and learn to distinguish and connect them. During this process, they come to see more clearly the differences and similarities between numbering and counting and between determining numbers. Children learn to recognize and count small numbers of items and are eventually able to make reasonable estimates, to arrange numbers and to compare them as “more”, “less”, “the same” and so on. Qualitative terms such as “very many”, “lots”, “enough” and “too many” are used to denote larger quantities.

Later, they become able to carry out simple addition and subtraction operations without having to actually see the relevant objects. Counting plays a crucial role in the development of an elementary sense of number. It is particularly the handy and flexible ability to count quantities in different situations that forms a major part of the foundation for the beginnings of the ability to calculate. It should therefore emphatically be involved in the early teaching in the kindergarten years. This may occur in spontaneous, response-improvised, or planned teaching activities, whether taking place in circle time, in the work corner, or in play activities. What is characteristic—and indeed essential—for the teaching approach that is reflected here is that the mathematical orientation takes place within a rich learning environment. This environment needs to connect with the world in which young children live and with their natural curiosity and inquisitiveness. The following section contains a brief description of how this can be brought about in teaching in kindergarten and how, by doing this, three levels of elementary number sense are negotiated.

The previous chapter showed that these experiences lead to the emergence of autonomous knowledge elements but it also described how this diversity of “number representations” for children can lead to great confusion.

A goose board game is being played in the gym.
Diana is counted off third and gets the number three. When it is her turn, she throws a four and says: “That’s not right. I’m number three!” Her friend then says: “You’re not three, you’re five.” (On asking, it appears that she means “five years old”).
Learning to count

Counting plus counting is two.
First of all, counting can consist of reciting a rhyme. This is a word sequence that can occur in a song or accompany a rhythmic movement or it can be a cue for a particular action. This so-called “acoustic” counting occurs in walking down stairs, in playing hide-and-seek and in other forms of counting out loud. Secondly, the action of counting serves to determine the sizes and numbers of things. The latter is called resultative counting and forms the basis for calculation. Counting and calculating initially constitute a single entity to such an extent that the first calculation problems are solved mainly by counting. However, children should eventually be able to make the transition from calculating by counting to learning to use number tricks and properties of operations. This, however, is not relevant here because of the primary focus on learning to count.

Many young children practice reciting the counting sequence at home. But there are also children who get little or no counting practice at home or who learn a non-Dutch counting sequence. There is therefore good reason to practice a varied reciting of the counting sequence in kindergarten with rhythmic movements, rhymes and songs, and various games.

Until one gets to the number fifteen there is no system in the Dutch names given to numbers, thus acquiring the sound sequence so that it can be recalled is purely a matter of imprinting through games and lengthy practice. For numbers from fifteen upwards, however, a pattern can be discerned in the number sequence. Some children discover the “tens” pattern by themselves, but most are not able to do so and this is not, in fact, necessary. This is because counting pays off mainly when it can be applied to determining quantities and sizes, and for comparing, adding and subtracting quantities. In other words, when resultative counting has developed. Until that happens, counting up to ten is more than sufficient.

The children know the counting sequence, at least up to ten.

The children get the opportunity to achieve this goal if, during teaching, sufficient attention is paid to counting, singing and moving games and to a varied reciting of the counting sequence, for instance by counting whilst “tagging” or counting out a row of items. All this should occur in natural and meaningful situations, possibly during discussion in circle time or in the play area.
Learning to count-and-calculate

Resultative and acoustic counting do not necessarily develop at the same rate. Initially, there is a considerable difference between how far children can recite along the counting sequence and the number of items they can count reliably. Evidently, resultative counting leads to specific conceptual problems. For instance, the counting of quantities greater than five—which cannot be taken in at a glance—often does not proceed at the same pace as the one-at-a-time pointing out of the relevant items. Sometimes the children count double and sometimes they miss an item. Moreover, the last number to be mentioned does not always function as a representation of the quantity. In such cases, in response to a repeated “How many?” question they run through the whole counting sequence again. Another possibility is to hold on to the last-given number as the label number, as Moira does.

“How many sandwiches are there?”
Moira: “One, two, three.”
“Eat one of them. How many are left?”
Moira: “Three.”
“But you’ve just eaten one of them.”
Moira: “Yes, but two and three are left.”

Different “results” do not lead to a conflict situation that could prompt the child to think through the counting activity more deeply. This is at first particularly common where the relatively larger numbers—above about five—are involved and tends to become more manifest in situations where the objects being counted are not neatly laid out or are only partly visible.

Questions of the “How many” type are, for young children who display this behavior, often only vaguely connected with indicative counting. This leads initially to an incomplete imitation of the outward counting action they see adults using. “How many” questions about relatively large numbers can apparently only be answered well by children who have already acquired a certain elementary number sense. In other words, children do not acquire a number sense just by having “How many” questions put to them and then being asked to emulate the concrete, synchronous, one-by-one counting activity. Teaching needs to offer more than that. How can teaching give extra didactical value?

Rikash (5;1) cannot yet do resultative counting. He counts a row of sweets non-synchronously and answers the question “If there are seven sweets in a tin and I take one out, how many are left?” with “Seven.”

Apparently, Rikash is thinking in the same way as Moira, Sweets number two, three ... seven are still in the tin—therefore seven.

How different is the situation when Rikash is playing a racing game over two parallel tracks. In this game, which is played with a die, he counts correctly and moves in time. How can this be explained? The answer is that counting in the context of the race track game is evidently meaningful to Rikash, namely in order to play fairly. But the pure counting of sweets is not yet relevant—let alone the pure, abstract, not-physical number...

In general, an elementary number sense develops at the pre-school level along three broad levels:
– the level of context-bound counting-and-calculating;
– the level of object-bound counting-and-calculating;
– the level of pure counting-and-calculating.
One could say that a child performs at a certain level if he or she works at that level in most cases, involving relatively difficult examples. These levels are explained in the following section.
The extra didactical value of context-bound counting-and-calculating lies in the specific nature of the “How many?” questions. Or, to be more precise, it lies in not posing such “bare” questions directly, but rather by putting them across in such a way that they are embedded in the child’s world of experience—in this case, in context—and can even emerge naturally from the situation. Take, for example, Rikash’s racing game. In this situation, non-synchronous counting is not acceptable. Therefore this way of counting is corrected in the natural situation of the game in which numbers are not explicitly asked for. Instead, the throw of the die serves to move the game forward by means of a convergent moving-and-counting action. And all this takes place in the social context of fair play. There is in fact more going on than this, as the natural questions about more, less, behind, quantity (more-less) indicate.

Context-based counting-and-calculating (level 1) takes place in meaningful, problem-related situations in which “How many?” and comparison questions can be put in an appropriate form. The ordinary context-free “How many?” questions of level 2 would not yet be understood and would be answered wrongly—and certainly if they involved relatively large, unordered, or poorly set-out quantities.

Asynchronous counting or counting past the end will therefore not occur so easily in such situations. Variable answers are no longer appropriate, while for object-bound counting it is not so important, at least from the point of view of the children, who treat it purely as a game. Luckily, opportunities for such meaningful, context-bound counting and comparing are widely available. Apart from the “How far?” question with the die and the race tracks and Ludo boards and the “How many years old?” question, there are many others.

Seven candles in a circle are hard to quantify and young children often count them asynchronously, do not watch out for start and finish, count on past the end, and so on. But in the context of a birthday party in which the candles on a cake are lit one after the other, questions like “How old?” or “How many years old?” become meaningful.

- how much does it cost? (in the shop corner prices are indicated by means of dot patterns)
- what time is it? (with imitated strikes of the clock)
- how high is the tower? (using stacks of building blocks)
- how many children are in the group? (in order to be sure that one has enough sweets to treat the children)
... and innumerable “How many?” and comparison questions in meaningful contexts about the number of nights slept, about ages, about estimating, and recounting small quantities.

In this way, other things besides counting quantities, in whatever form, are brought into play. Operations such as ordering, comparing, estimating, “add”, “take away”, “more”, “less” and “as many” can also be introduced. Turning this the other way round, the teacher can create situations in which the children count synchronously but where the connection with resultative counting (determining a quantity) does not have to be readily apparent. This happens with numbering in order, as in “one, two, three, four, five ...”

In a circle of twelve children, each gets a telephone number to remember. One child gets the task of assigning the numbers in sequence in the circle. The child can decide for himself where to start and thus who gets number one. Does the child forget to give himself a number? Then number six is called, stands up and has a short conversation with the caller. “Good. Number nine ...?” Nothing happens. Some children start counting from the first child. Others count on. The children on either side of number nine suddenly know who has to stand up: Felix!

Then it is the turn of another child to hand out the numbers, except that indicative numbering, of course, should start with a different child this time. The sequence ends with twelve again! Why’s that? Felix now pays attention: “Of course it’s twelve; no, one’s gone away!”

In the next round, one child does leave the circle but the numbering starts with the same child again. Which children keep the same number and who gets a different one this time? How many numbers do we need now? “One less, because from there everything moves one place up.”

Notice how here the giving of a number name and the resultative counting can be connected in the context situation described for the children in a circle.

2

Within what are, for them, meaningful context situations, children are able to count to at least ten, arrange numbers in the correct order, make reasonable estimates, and compare quantities as being more, less or equal (level 1).

The connection and integration of the comparatively separate knowledge elements of the ground level—and thus the transition to level 1—can be stimulated by activities such as those described for the racing game, the birthday party and shopping. Through these activities, synchronous counting acquires a meaningful function in making honest moves, lighting the candles on the birthday cake one-by-one, and buying presents in the shop corner, respectively. In such cases, synchronous counting is linked—just as with numbering—in an implicit and natural way with the determination of quantities. Teachers can pay sufficient attention to this context-bound counting-and-calculating through spontaneous, prepared and improvised activities in circle time and in the activity areas.
Object-bound counting-and-calculating (level 2) occurs in problem situations that are focused directly on the quantitative aspect. In contrast to the first level, directly posed “How many?” questions are understood at this level. This holds true, however, only in so far as the questions relate to concrete objects, that is, whenever physical numbers are involved: “How many sweets are in the tin?”; “How many candles are there?”; “How many blocks are in the stack?”; “Which tin has the most sweets in it?”; “How many of these seven candles remain alight if three get blown out?”

At the level of object-bound counting-and-calculating the skilful organization of counting plays a central role—the children have to lay out the objects in a neat pattern to get a better grip on the counting.

The first didactical question is now, of course, how best to assist the transition from the context-bound level 1 to the object-bound level 2? The answer seems to lie in gradually pushing the context into the background. In the birthday party context, for example, attention can be shifted during the course of the school year by changing the question from “How old?” to “How many candles?” These candles could then be associated with a row of candles on the birthday cake. Counting unordered objects in an orderly way by placing them in succession or in a neat pattern then emerges naturally from the context. The same can be achieved by shifting the “How high?” question about the stack of building blocks to a “How many blocks?” question, or from how many children there are in the child’s own group to how many there are in any particular group, or from dots that indicate prices to the “pure” counting of dots, and so on.

The didactical benefit of gradually shifting the nature of the question is that the original context-related situation remains, in principle, available for those children who still need it. In this way, the context functions as a model within which these children are able to gain the basis they need in order to be able to interpret and answer the “How much?” question, leading to synchronous and resultative counting-and-calculating with concrete objects. Once the children have reached level 2, they can practice a varied and wide range of counting-and-calculating activities. Take, for example, the following calculation puzzles about concealed objects.

- How many blocks lie under the cloth? Briefly raise the cloth then ask for a reasonable estimate. Who’s got it right? Check by counting the blocks one after the other as they are pushed aside. Vary the number of blocks up to about ten. With up to about five objects, the number is grasped at a glance (level 1). Children then know where their counting has to get to. This then also helps in the synchronous counting of larger numbers of objects.

- If the number is known, say “seven”, one block is taken from under the cloth. How many are still under it now? How did you work that out? Who has the right answer? Check by re-counting. Then another block is placed next to the cloth. How many are under the cloth now? Are there more blocks next to the cloth or under it? And so on.

- The concealed object games can also be made more complex, as in the following account with Kim (5;5).

  Kim: “Seven.”
  Teacher: “Now I take one away. How many are still inside?”
  Kim: “Six.”
  Teacher: “Now I take another one out.”
  Kim: “Five.”
  Teacher: “Good. And now another.”
  Kim: “Four.”
  Teacher: “Now I’m going to make it harder. I take two more out. Now how many are still inside?”
  Kim: “Two.”

- Addition problems can also be done with concealed object games. Three blocks are visible and four are
under the cloth (or in the box). How many blocks are there in total?

Addition and subtraction can be practiced by lighting and blowing out candles, by the addition and removal of objects and with concealment puzzles. The act of making objects invisible by concealment specifically marks the transition from level 2 to level 3 (more about this shortly). There is more going on at level 2 than simple counting, however: adding to, taking away, ordering, linking, coupling, comparing, and making a reasonable estimate of a number of objects are all core activities that are practiced in the number range up to ten. Measuring can also be introduced by, for example, having children pick up boxes of large marbles to compare the weights, guessing how many marbles are in the box and then, finally, seeing whether they guessed right. Weighing and counting go hand in hand.

Children can order, compare, estimate, and count up to ten objects. They are also able to select a suitable strategy for simple addition or subtraction situations in such things as concealment games for up to ten objects (level 2)

The children have the opportunity to reach these goals if teachers reserve sufficient time for estimation, counting and calculation puzzles with concrete objects that are either visible or where some are covered up and therefore have to be imagined.

For example:

- Make a reasonable estimate (up to ten) of the number of sweets in a tin and then count them. Did I guess too many or not enough?
- Then solve “add one” and “take one away” problems in the head: “Take seven sweets. Add one makes …” “Having seven sweets and taking one away leaves …” Then do it with real sweets and check the result.

The starting numbers and the numbers added or taken away should also be varied.

Teachers should also bear in mind that the counting of up to four or five objects differs fundamentally from counting larger numbers of objects that cannot be taken in at a glance. It also makes a difference whether the objects are shown disordered, in a row, or in a recognizable pattern. The calculation problems and puzzles should be presented in an exciting context with a variety of narrative and imagery. The children should be given the chance to exchange their different counting strategies.
Towards pure counting-and-calculating via symbolization

With the pure counting-and-calculating at level 3, a question like “What is seven take away three” is understood and worked out correctly by, for example, using fingers. By concealing the objects (level 2), the children are obliged to use fingers or other representations for more complex tasks. In this way, counting ceases to be object-bound and is, instead, transferred to physical or mental representations of the objects. These representations can occupy very different levels of abstraction, including that of “pure” arithmetical numbers. Take, for instance, the task of showing one’s age within the birthday theme. This can be done with a “birthday hat” but there are other possibilities. A nice assignment is, for instance, to ask the children to express their age without using words.

Apart from the row of candles on the birthday cake, the children represent the idea of “five years” in the following ways:
- showing five fingers on a hand
- throwing a five with a die
- laying out five building blocks
- drawing five tallies
- pointing at the number 5 symbol.

The first four representations contain “five” in a countable form. In contrast, the number “5” symbol in the last example, while certainly referring to five countable things, does not contain a visible or countable quantity. Countable number representations are abstractions of named numbers at level 2, and concretizations of pure numbers at level 3.

As such, they perform an essential bridging function between concrete and formal calculation. This is why working with imagined but non-visible quantities in “guess and cover” tasks is so important. Practicing with the two-sided numeral or dot-pattern cards often used in the shop corner can reinforce the link between the countable representation and the relevant numeral.

This discussion about years 1 and 2 of kindergarten concludes with the example of Fadoua.

The teacher is presenting Fadoua (4;10) twelve number cards in a random order and asks her to order them. Fadoua names the number cards from 1 to 12, and lays them out in order.

When she has finished, the teacher asks her to close her eyes. He takes away card with the number seven and says: “Look at what’s happened.”

Fadoua looks and says: “It’s eleven.”

She sees that one card has been removed from the row and turned over.

She points to twelve—the last card in the row—and says: “Eleven.”

By this she means that there are still eleven cards in the row. Having the number twelve at the end does not confuse her!

Fadoua is, in fact, ready for the pure counting-and-calculating of level 3: “Twelve take away one is …” But the activities at this level do not belong to the core activities of kindergarten, although there will certainly be children who can cope with this sort of formal manipulation, and even structured calculation, once they enter grade 1 of primary school. They solve level 1 and level 2 problems at a higher level and that is no bad thing. On the contrary, this differentiation of levels is inherent in good mathematics teaching, and if these children can then be encouraged to put their solutions into words, the other children will be able to take advantage of it.
Children can represent physical numbers up to ten on their fingers and with lines and dots, and are able to use these skills for “adding up” and “taking away.”

In the teaching, attention should be given to the various levels at which elementary counting-and-calculating tasks can be introduced and solved. The teacher should recognize the importance of symbolization with countable characters or with numerals in making the transition to the pure counting-and-calculating situation of level 3. (The use of numerals and doing simple arithmetical operations with them at level 3 is not really one of the objectives for the end of year K2, but there will usually be children who can work at this level.) Teachers should ensure that the children are given the opportunity to work at different levels and are sufficiently challenged to solve counting-and-calculating tasks with whole numbers at a level in keeping with their development. At the same time, there should also be enough room for the children to interact with each other as well as with the teacher, and with due attention being given to the language aspect.

Sketch of the learning-teaching trajectory

Of course, not all children work their way through the levels described for counting-and-calculating in the same fashion or at the same time. Children should be given all the space they need to progress in their own way, particularly in the first year. Considerable differences can show up within each group and the teachers may find themselves having to deal with all three levels at once within the same teaching situation, as the following example shows:

The class sit in a circle and some building blocks are stacked up in different ways in the middle. The children then decide how many blocks (“rooms”) there are in each structure (“building”). The answers can vary greatly, especially for the larger structures.

For the structure shown above, for example, the answers were five, seven, eight, twelve, fifteen, and even sixteen and seventeen.
Teacher: “Well, isn’t that odd, so many different answers?”
For some children, this is not in the least bit odd. For them it's just one of the things that happen with counting—it's a game in which you sometimes come out with five and sometimes get eight, and it really doesn't matter which. But other children think differently. Fiona comes forward and
shows how she counts. She starts at the base and counts all the visible faces: “One, two, three, four…” and ends up at the top having counted fifteen.

Others: “It’s much less.” “She’s counting double.” “You don’t count like that!”

Then Rico comes forward. He counts the blocks correctly from the top down: “One, two, three, four….” The blocks in the lower layers he counts by their faces: “Five, six, seven, eight, nine … twelve.” As he counts, his voice becomes increasingly uncertain and when he’s finished he says straight away: “Hey, that’s odd. My mother says that four and four is eight, and now I’ve got twelve!”

Other children join in the discussion. When Robert gets his turn, he asks: “Can I take them apart?” He takes the structure apart and counts the blocks one by one while more and more of the children join him in counting...

Everyone is now satisfied that there are indeed eight blocks.

When the structure is reassembled, the teacher has a final question: “How did you see that it was four and four, Rico?”

Rico answers: “Well, this is four (he takes the top four blocks off) and this is, too.”

The blocks are counted again and there are indeed eight. Four and four is eight.

All three levels of counting-and-calculating can be found in this scenario. The level of context-bound counting-and-calculating was found amongst those children who do not consider the “How many?” question to be all that important in this situation. They are not at all surprised that counting gives different results at different times. Then the level of object-bound counting-and-calculating shows up in Robert’s approach, who makes the counting more manageable by separating the two layers of blocks for a moment. Finally, Rico used pure counting-and-calculating and just missed working it out: it’s four and four and that’s eight. Looked at in this way, this cross-section of the different approaches used by the children also shows how far along the learning trajectory they have traveled.

It is precisely through picking up on the children’s different approaches and then discussing them in the class that the richness of teaching can manifest itself. What’s more, the children have been able to learn useful approaches from one another. They have also become increasingly aware of the importance of organizing counting in an easy way and, in doing so, have become more inventive in getting to grips with counting.

The teacher has the job of distinguishing the various levels of thought and action, of recognizing and building on critical points, and of encouraging the use of particularly fruitful approaches. Reflective questions such as that put to Rico (“How did you see that it was four and four?”), show how an extra impulse can be given to such opportunities for exchange. The result of all this is that the children eventually, and each in his or her own way, begin to feel more at home with handy and flexible counting, thus laying much of the basis for later addition and subtraction skills.
Summary

By the end of the second year of kindergarten, the children should be able to count to at least ten and should be able to use this skill for making reasonable estimates, for ordering, comparing and determining, as well as for manipulating whole number quantities. They should, for example, be able to solve the puzzles set out in the rhyme below by calculation as well as by rhyming, or be able to check the result by using various counting procedures—provided that the sweets mentioned in the rhyme are actually present.

**Counting by rhyme**

Seven sugary sweets sitting in a jar,
but one was crushed between two bricks. Now there are...
Six sugary sweets. Then came an old man’s wife who took one away. Then there were just...
Five sugary sweets. Then came my niece Marie, who took two. Then there were just...

Three sugary sweets. Then came the man from the store, who brought me a sweet.
Then there were...
Four sugary sweets, and then came Auntie Gwen. She put six sweets in the jar. And then there were...

Ten sugary sweets. I ate them all myself. Now the jar is empty sitting on the shelf.

(Original Dutch version by Annie M.G. Schmidt)

The following classroom observation starts with the line “Four sugary sweets … put six sweets in the jar.” It shows two main strategies.

There is a drawing of a glass jar in the middle of the circle. Some children solve the problem by counting all the sweets and put them in the jar, while others count on from four or six.

This latter strategy is harder to accomplish because it involves doing two things at once: counting-on while remembering how many one has already counted so as to produce a running total.

Both strategies can also be practiced with representations rather than real sweets, e.g. with fingers, lines, beads or counters. Representing quantities by various countable images is also one of the objectives for kindergarten.

How many sweets are there? Whether young children in kindergarten can answer a question like this will depend on:

- the number of sweets, whether there are more than six (five)
- the spatial form in which the sweets are arranged—in a row, in a regular pattern as on a die, or in a random heap
- whether or not the sweets are visually or physically present
- the way in which the “How many?” question is put—either directly as “How many sweets are there?” or indirectly as “How many children could you give a sweet to?”

Indirect, context-bound “How many?” questions regarding small quantities of visible objects arranged in a neat pattern are the first to be understood and correctly answered (level 1). Questions about estimating, ordering, comparing, “adding to” and “taking away” that have these characteristics are also meaningful at this level.

Straightforward “How many?” questions about quantities of six or more objects that have not been neatly laid out will still initially present many children with insurmountable problems—they count asynchronously, count on past the answer, and so on.
A number of didactical routes are available for easing the transition to the object-bound “bare” numbers at level 2:
– starting with direct but easily grasped “How many?” questions about small quantities (up to four) that can be taken in at a glance, having the children check the answers, and then gradually introducing larger numbers
– using indirect context-bound “How many?” questions about larger quantities and then gradually pushing the context into the background while putting the questions more directly.

The same applies to questions about estimating, ordering, comparing, “adding to” and “taking way.”

The transition to calculating operations at level 3 is stimulated by questions about estimation, ordering, comparison, “adding to” and “taking way” without using visible sweets (or other physical objects), which urges the children to work with self-conceived representations.

The task of working out “four sugary sweets plus six sugary sweets” without any visibly present sweets gives the children a lot to do, but this is justified by the results, namely pure numbers that can be conceived concretely in different ways and which can be worked with in several ways.

The above attainment targets of counting and elementary calculation (making a reasonable estimate, ordering, comparing, “adding to” and “taking away”) with physical numbers that refer to concrete objects or measurements are put into a teaching framework.

This framework distinguishes five aspects of number and three developmental levels of elementary number sense. The targets are pursued with the help of spontaneous, improvised, and prepared teaching activities in interactive group teaching using a variety of didactical methods, as well as in the more individual settings of the work corners.

In this way, every child counts …

In kindergarten, the counting sequence is extended further. The children gradually develop an understanding of the various meanings and functions of small numbers and the relationships between them. During this process, they learn to recognize numerals. This expansion of the world of numbers is called “emergent numeracy.” Attention is paid in teaching to the different aspects of numbers and the teacher has insight into the different levels within which an elementary number sense develops.

This development can be stimulated by a combination of spontaneous, improvised, and prepared teaching activities. Mathematics education in K1 and K2 often has a strongly integrated character, which makes it even more important to watch out for those isolated “golden moments” of opportunity. This sketch of the beginnings of the learning trajectory for whole numbers aims to give a clear view on this emergent numeracy and, in doing so, to promote the development of elementary number sense.