Young Children Learn Measurement and Geometry
A Learning-Teaching Trajectory with Intermediate Attainment Targets for the Lower Grades in Primary School

Marja van den Heuvel-Panhuizen and Kees Buys (Eds.)

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FOR
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TAL Project
Freudenthal Institute, Utrecht University
National Institute for Curriculum Development (SLO)
in collaboration with CED Rotterdam

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You will need to sign in with username cd-rom and password video
Foreword

Toward education of high quality
Improving the quality of education is an important ambition of educational policy. The TAL project aims to contribute to this. It is a project initiated by the Dutch Ministry of Education, Culture and Science, and carried out by the Freudenthal Institute (FI) of Utrecht University and the Dutch National Institute for Curriculum Development (SLO), and partly conducted in cooperation with the Rotterdam Center for Educational Services (CED). The quality of education can be improved in many ways. TAL proposes to do this by providing insights into the broad longitudinal outline of the learning-teaching process and its internal coherence. The intention of TAL is to give support to teachers in combination with the guidance they get from mathematics textbook series.

TAL
The TAL project aims to describe the intermediate attainment targets of primary school mathematics, covering kindergarten through grade 6. These targets are a further elaboration of the core goals for the end of primary school. A characteristic of the intermediate attainment targets is that they are embedded in a learning-teaching trajectory. This is also the reason for calling the project TAL, which in Dutch stands for Tussendoelen Annex Leerlijnen; in English this means intermediate attainment targets in learning-teaching trajectories. Instead of Annex, the middle letter of TAL can also be considered as referring to Afbeeldingen (Representations). The latter interpretation indicates that the trajectory description contains many examples of students’ behavior and teaching activities. These examples form an essential part of the learning-teaching trajectory. The TAL learning-teaching trajectories comprise the main subdomains of mathematics in primary school. Young Children Learn Measurement and Geometry contains the learning-teaching trajectory for measurement and geometry in kindergarten 1 and 2 and grades 1 and 2. After the earlier published learning-teaching trajectories for calculation with whole numbers in the lower grades of primary school and the higher grades of primary school (Children Learn Mathematics), this is the third TAL learning-teaching trajectory.
With this new trajectory the mathematics program for the first four years of primary school has been completed. For the higher grades of primary school further development work is needed; among other things, a continuation of this learning-teaching trajectory for measurement and geometry.

**The work of many**

In addition to the TAL team, many people contributed to the creation of this learning-teaching trajectory for measurement and geometry in the lower grades of primary school. The comments of experts who took part in the consultations² have been of great value for the choices that were made. Their commentary and additions have enriched and improved the learning-teaching trajectory. Moreover, as a result of the consultations, the learning-teaching trajectory was able to obtain wide support. A special thanks goes out to the schools that were involved in the development of this learning-teaching trajectory.³ Thanks to the teachers’ input and the cooperation they provided in trying out the teaching activities and in making the video recordings, the learning-teaching trajectory description has become what it is today. The Advisory Committee⁴ provided a communal policy for the learning-teaching trajectory descriptions of the subjects involved. Moreover, this committee has opened doors for the implementation of TAL, through the development of information modules for teachers and in-service training modules for mathematics coordinators.⁵

**Learning-teaching trajectories as a source of inspiration**

We hope that the learning-teaching trajectory description for measurement and geometry succeeds in stimulating classroom practice, and that it inspires teachers to didactical efforts on a high level, in what was up to now, in the lower grades of primary school, a less-known subdomain of mathematics. The learning-teaching trajectory with intermediate attainment targets offers support to teachers, in order to give measurement and geometry a full and worthy place within the mathematics curriculum. For that to be the case, the foundation that is made with this learning-teaching trajectory must be built upon in the higher grades of primary school and beyond.

TAL team, Utrecht, the Netherlands, April 2005
Completion of the learning-teaching trajectory for mathematics in the lower grades

The learning-teaching trajectory for measurement and geometry in the lower grades of primary school shows how children develop an understanding of their physical surroundings through mathematical tools and insights. A view of how this process takes place is required in order to provide the children with an adequate learning environment and to make appropriate didactical decisions.

The now published learning-teaching trajectory is an addition to the learning-teaching trajectory for calculation with whole numbers in kindergarten 1 and 2 through grades 1 and 2. With this volume, the learning-teaching trajectory for the subject of mathematics in the lower grades of primary school is completed.

As opposed to what is usually thought, mathematics in primary school includes more than just arithmetic. Mathematics is not only connected to the world of numbers, but also to the physical world around us and the phenomena that occur within it.

In geometry, the issue is to understand the space around us in the broadest sense of the word. It is related to the two- and three-dimensional world and the related shapes and figures.

Measurement, on the other hand, is aimed at quantifying our physical environment. The emphasis in this process, which is on describing the world in measuring numbers, makes measurement the connecting link between arithmetic and geometry.
The publication of this learning-teaching trajectory does not mean that, in the Netherlands, the subject of mathematics in primary school is now increased with two new components. Measurement has a long tradition in Dutch primary school teaching – albeit not always with a “realistic” foundation.3

Geometry for primary school has a shorter history, but it cannot be called “new” either. Actually, “realistic” geometry has been part of the primary school curriculum since establishing the core goals in mathematics in 1993.4

Apart from the fact that both domains are mentioned in the core goals5, the general goals of mathematics education in particular show how essential measurement and geometry are for children’s development.

Mathematics education is aimed at children:
– being able to make connections between mathematics and their daily environment
– acquiring basic skills, understanding simple mathematical language and applying it in practical situations
– reflecting on their own mathematical activities and checking results
– searching for simple connections, rules, patterns and structures
– describing investigative and reasoning strategies in their own words and using these strategies.6

Both measurement and geometry enable children to make connections with their daily environment. Both domains offer mathematical tools, each in their own way, to structure the physical world and to get a grasp on it. Moreover, they both lead to wonderment, and thus to the development of a mathematical disposition which is characterized by an exploring attitude, a certain perseverance in solving problems, and a sensitivity to the beauty of mathematical structures and solutions.
Once per week either measurement or geometry
Assuming that one mathematics lesson is taught every day – meaning five lessons a week – this TAL book covers mathematics in kindergarten 1 and 2 through grades 1 and 2 for one day a week; one week a measurement lesson is taught, and the next week a geometry lesson. Naturally, this is a very general schedule. Moreover, the term “lesson” should not be taken too literally. Especially in kindergarten 1 and 2, activities in the fields of measurement and geometry often form a natural component of the experienced-based and development-focused approach to teaching, and they often spontaneously become part of other activities.

Intermediate attainment targets and teaching frameworks
This TAL book provides an overview, through the use of core activities, of the pathway followed by the student in the lower grades of primary school in measurement and geometry. The crucial learning steps the students take in this process are formulated as intermediate attainment targets. In connection to this, in teaching frameworks, a summary of how teaching can contribute to reaching these intermediate attainment targets has been included. The intermediate attainment targets and the teaching frameworks can be recognized as follows.

<table>
<thead>
<tr>
<th>Intermediate attainment target</th>
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<table>
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<tr>
<th>Teaching framework</th>
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A learning-teaching trajectory with classroom examples
Characteristic of this learning-teaching trajectory description is the inclusion of classroom examples. They are meant as clarification of crucial elements in the learning-teaching trajectory, and they show in a concrete way the type of didactics that we have in mind. In a large number of these classroom examples, a camera is depicted.

This means that a video recording was made of this passage. The clips are available for download. In addition to video clips, the downloadable material also includes a demonstration version of the computer program “Building with blocks.”

The learning-teaching trajectories for measurement and geometry
The trajectories for measurement and geometry are laid down in two separate sections in this book.
The first section, on measurement, is comprised of these chapters:
- Domain description measurement
- Measurement in kindergarten 1 and 2
- Measurement in grades 1 and 2
The second section, on geometry, is comprised of these chapters:
- Domain description geometry
- Geometry in kindergarten 1 and 2
- Geometry in grades 1 and 2
- Geometry on the computer in grades 1 and 2.

Although the learning-teaching trajectories for measurement and geometry – just like those for other domains in mathematics – cannot actually thought of as distinct from one another, they are still described separately. In this process, the same procedure was used as in the domain of calculation with whole numbers, in which the various forms of calculation were also put in separate strands. Creating clarity in the structure of a learning-teaching trajectory requires certain limitations. Only then is it possible to expose the specifics of learning and teaching measurement and geometry.
Two domains with their own characteristics

Measurement and geometry are two domains, each with their own nature, which can be easily recognized in the learning-teaching trajectories. The learning-teaching trajectory for measurement, for example, is more linear in its structure than the one for geometry. When it comes to that, measurement is closer to calculation with whole numbers, where the size of the numbers provides a natural sequence in the learning process at hand. In a comparable way, more and less basic forms can be discerned in measurement as well. For example, length precedes other physical quantities.

In geometry, on the other hand, there is less of a linear structure. One cannot say, for example, that determining the position of a photographer comes before or after reflecting shapes and figures. These are skills that appeal to various aspects of geometry, the development of which often takes place simultaneously. That is why the learning-teaching trajectory for geometry has a more cyclical character. The various aspects of geometry keep returning each time at a higher level.

Apart from the fact that the trajectories for measurement and geometry are different, their intermediate attainment targets also vary in nature.

Whereas for the kindergarten classes, measurement attainment targets are formulated that indicate what the children should be able to do, the emphasis in the geometry attainment targets lies in providing a rich foundation of experience.

An important point, which should not be missed in the characterization of the two learning-teaching trajectories, is that measurement is didactically more crystallized than geometry. The measurement trajectory that is set out now is, in fact, a reconsideration of a rich domain-specific didactical tradition. Geometry in primary school, on the other hand, is still relatively unexplored in the Netherlands. The learning-teaching trajectory that has now been developed, is in fact the first outline that has been made for this level in geometry teaching. This does not mean that nothing has been done in this area thus far.

The more recent versions of the mathematics textbook series contain a number of geometry activities. However, these activities do not form a coherent structure. Also, the textbook series differ quite a lot in the choice of geometry topics that they address.10
The goal of TAL is to change this and to achieve more coherence in the teaching activities and learning content.

Since 1997, when the TAL project started, the concept of the learning-teaching trajectory has been more and more firmly established in education. However, this does not mean that there are no more misunderstandings. Still it is necessary to emphasize that the TAL learning-teaching trajectories are not meant to prescribe the learning pathway of individual students in a step-by-step way. Neither are they a replacement for the existing mathematics textbook series. The learning-teaching trajectories describe the teaching-learning process on a more abstract level. They should be regarded as a description of an idealized classroom practice, which at the same time and as a result of many examples, is rooted in everyday teaching. The learning-teaching trajectories are meant to help teachers in developing a theoretical plan, a conceptual frame of reference that the teacher has in her/his mind, with which one can continually make all kinds of didactical decisions. The learning-teaching trajectory offers the teacher a macro perspective for that purpose, a view of the longitudinal process. The learning-teaching trajectory answers the question of how certain skills and insights can be built up over a number of years, and which content choices have to be made in that process. In the TAL trajectories, these choices were made, but the result is definitely not a straightjacket. It should also be said that the trajectories were not developed in a top-down way. They were created with the help of many, through research, study of literature and exchange of ideas, and they reflect the current state of our thinking about measurement and geometry education.

The discussion is in no way closed. The TAL learning-teaching trajectories are meant to put the choices of teaching activities and learning content in primary school on the agenda, both in schools and on a policy level.
Measurement as a phenomenon

Measurement in a historical perspective
Measuring can be described as ordering our surrounding world through numbers, in order to better control that world. With this goal in mind, numerous measuring methods have been developed in the course of history, methods that were not only aimed at practical, everyday matters, such as measuring quantities precisely, but also more important matters like determining the circumference of the earth and determining the correct longitude at sea. Moreover, since the start of the last century, the human measuring activity has no longer remained limited to phenomena from the natural sciences. Social, psychological and societal problems have become more and more the object of research, with the help of measuring methods.
We are confronted almost daily with its results, in the form of population prognoses, pollution scores, poverty data, IQ scores, and so forth.
Many of the measurement results in question are presented through surveys, schedules and charts, the contents of which are highly determinative for important social developments or decisions. As a result, measurement has been gaining more and more importance in our society.
In connection with this, measurement comprises an increasingly important subject matter area in mathematics education.

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Weight increase for primary school children
Percentage overweight* children

*overweight is defined as exceeding internationally agreed maximum body mass index (differs per age up to 18 years)
Measurement in the lower grades of primary school

On an elementary level, measurement is often initially associated with the use of measuring instruments like the folding ruler, measuring jug or the kitchen scale. Through reading a number off of a measure scale, the length, the volume or the weight of an object is determined. Of course, this is an important kind of measuring, but it is not the only one and certainly not the most elementary kind that can be dealt with in education. Measurement should comprise more.

For example, measuring a distance by pacing it off with big steps is also a form of measuring. The same goes for counting out a cup of rice per person in the preparation of a meal, and for measuring off a time span by counting in a game of hide and seek. In such situations, measuring takes place through pacing off, during which a certain obvious measure is used to express globally a distance, a volume or a time period.

Naturally, this is a less accurate, but in some cases certainly effective form of measuring compared to measuring with a standard measure.

However, there is still an even more elementary form of measuring, namely the form in which measuring takes place by comparing two or more objects directly in length, weight, volume, area or time.

This most elementary kind of measurement presents itself often in the world of a young child. This is seen, for example, when two children compare who has the bigger piece of licorice, the bigger cookie or the fuller glass of lemonade. In this kind of measuring, no numbers are involved yet. The result of the measurement is that they verbally express the question of which of the two objects is bigger or heavier or has the greater volume. Such measurings do not always have to be so easy. For example, how do you determine which glass contains more lemonade? And how do you determine which one is the bigger of two differently-shaped cookies?
In short, there are three essential forms of measuring that are important for the lower grades of primary school:

– measuring through comparing and ordering
– measuring through pacing off, using a measurement unit (natural or standard measure)
– measuring through the use of a measuring instrument.

Of course, there are still other forms of measuring, like the one in which the measurement takes place predominantly through reasoning and calculating. For example: approximately how high is a building of six stories? And: in the newspaper, it said that in the province of Utrecht, the Netherlands, an average of over 50 millimeters of rain fell in 24 hours; approximately how many cubic meters of water is that in total? In such problems, the point is that the children retrieve missing information themselves, and that they think of an adequate reasoning strategy with which they can approach the outcome. Although this form of measuring can also play a role in the lower grades of primary school, it is still mainly reserved for the higher grades.² In the lower grades of primary school, it is mostly the three more basic forms of measuring mentioned above that are essential.

**Concept development and constructive input by children**

In education, these three forms can be dealt with consecutively and in such a way that the connections among them are made apparent. The basic point in this is not so much practicing measuring activities, but much more gaining insight in these activities and, on a more general level, developing a proper concept of measuring. For comparing and ordering, this can happen through problem situations in which children develop their own appropriate comparison strategy, as, for example, in the abovementioned case of comparing the volumes of two beakers, or as in the case of comparing a number of curvilinear objects, like the children’s own waist, or the circumference of a number of trees. Also, the attention regarding measuring through pacing off can be aimed at problem situations in which the children develop an appropriate measure, and investigate this pacing off. One example, for instance, is the situation of the *jeu de boules* game, when the distances from the balls thrown to the little ball must be compared.
A similar idea applies to measuring through working with a measuring instrument. It is primarily investigating how measuring in that way works and sometimes even developing such an instrument which is of great value. On a more general level, it is mainly the focus on concept development and on the children’s own constructive input which leads to valuable and appealing learning experiences in measuring, and thus has to be considered an essential aspect of the learning-teaching trajectory that is described hereafter.

**Several physical quantities to which measuring applies**

As was mentioned before, the term measuring applies to determining the size of objects or phenomena. Measuring provides answers, in a specific way, about a certain size characteristic of one or more objects or phenomena. These features can apply to the most basic physical quantities like the ones indicated above: the length (thickness, height, depth, …) of an object, its volume, its weight, the speed with which it moves, the time it takes to do so and the area it covers. Naturally, there are also other quantities like density, force, energy, luminosity and such, but these are too advanced for the lower grades of primary school.

Of the above mentioned physical quantities, length is undoubtedly the most primary. Not only is it available to the children’s perception, it is the most indicative quantity people want to find out about all sorts of objects.

But also measuring other physical quantities can often, in one way or another, be reduced to a form of length measuring. In the previously mentioned example of the two glasses of lemonade, this is already obvious. But it turns out to be even stronger in viewing all kinds of measuring instruments. A measure scale is often included in these instruments, on the basis of which a certain size characteristic like weight, volume or speed is measured by reading off a length of a measure scale.
It is therefore obvious to award a central position to activities in the field of length. Attention can also be given regularly to other basic physical quantities like volume, weight, time and speed. Obviously, the differences between all these physical quantities are initially far from clear to the children. They still see little difference between various size characteristics like length, volume, weight and area.

This is also expressed in the language: terms like big and small are often used as a general size indication, while it remains unclear which measurement is at hand. Partly under the influence of education, this is gradually changing. The children become increasingly aware of the characteristic features of the various physical quantities. Along with this awareness, the language use becomes more and more differentiated. Phrases like “that will fit more” and “they are almost equally heavy” emerge progressively, just as do the different ways of indicating length using terms such as long, thick, wide, high and so forth. The developing awareness of physical quantities, as well as the accompanying language development, form the basis for a more specific exploration of the other distinct physical quantities, as they are described in the following chapters.

A more qualitative exploration (what does weight mean, which terms are used with it, how can you weigh on your hands, how can you try to make yourself heavier?) precedes in this process before a more quantitative approach (what are conventional weight measures, how does a kitchen scale work, how do you use a letter scale?), while calculating and reasoning with measures (measure conversion, price-weight problems, volume and weight enlargement effects) are dealt with later.

**Why measurement?**

Why is measurement an important subject matter area in mathematics education? This topic has already been touched on in the previous sections. First of all, measurement comprises an aspect of practical skill that is important in daily life.
Measuring length with a ruler or folding ruler, measuring volume with a measuring jug, determining weight with a letter scale or bathroom scale – these are all forms of measuring that occur regularly in everyday life and also play a role in secondary education. It is therefore obvious that children should become acquainted with them during their primary education. In this process, the emphasis does not necessarily have to be on practicing extensively with a measuring instrument, but more on learning to fathom the principles on which this measuring is based, and on developing a level of familiarity with important aspects of measurement, such as estimating and interpreting results.

But there are more reasons for bringing up the importance of measurement. Knowledge of the most important measures and measure systems, as are used in real life, is also important. This is proven just by looking at the following chart, which is available at every post office in the Netherlands.

<table>
<thead>
<tr>
<th>Weight</th>
<th>Stamps</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 – 20 gr</td>
<td>0,39</td>
</tr>
<tr>
<td>20 – 50 gr</td>
<td>0,78</td>
</tr>
<tr>
<td>50 – 100 gr</td>
<td>1,17</td>
</tr>
<tr>
<td>100 – 250 gr</td>
<td>1,56</td>
</tr>
<tr>
<td>250 – 500 gr</td>
<td>2,25</td>
</tr>
<tr>
<td>500 – 2 kg</td>
<td>3,00</td>
</tr>
<tr>
<td>2 – 3 kg</td>
<td>3,00</td>
</tr>
</tbody>
</table>

Much knowledge of weight and length measures is needed to interpret this chart correctly and to use it. It is not only about the measures themselves (centimeter, gram, kilogram), but also about the connections between them and the coherence of the measure system of which they are all part. It is very valuable that in addition to the more practical measuring experience, children gain insight into the different measure systems, especially in the decimal structure of the metric system.

In connection with this, it is important that the children build a system of personal reference measures: a series of measures obtained from everyday reality, with which the size of all kinds of objects can be estimated, and, more importantly, which makes the order of size, of the various standard measures more easily imaginable.
For example:
- the width of a finger as a reference measure for the centimeter
- the thickness of a fingernail as a reference measure for the millimeter
- a big step as a reference measure for the meter
- a bottle of milk as a reference measure for the liter
- a pack of sugar as a reference measure for the kilogram.

There is another reason why measuring is an important subject.
As is described in the chapter on numbers and number relations in the TAL book in which the learning-teaching trajectory for calculation with whole numbers for the higher grades in primary school is explained³, the measurement aspect forms one of the central numerical aspects that are important for the acquisition of number concept. Measuring numbers represent a specific aspect, because they refer to an “environment” in which the number exists. For example, a measured distance of 2.45 m (or two meters and 45 centimeters, or 245 centimeters) refers to a distance of around 2.45 meters — it might be a bit more or a bit less, but not a lot more or less. Also, this distance may be interpreted as almost two and a half meters, as almost 250 centimeters, or even as two meters and four and a half decimeters.

> How far is it when you jump 2.45 meters? And what if somebody jumps 3.79 meters? Or 7.98 meters? Does that last jump fit in the classroom?

By giving the carrying out of physical quantities, like the one mentioned above, and the reflecting on them a place in education, children may be strongly encouraged to obtain better insight into the numbers, into their global position on the number line, into the environment the numbers are in, into the connection to surrounding benchmark numbers and such.
Also regarding other domains of mathematics, a good knowledge of measurement is important. For example, being comfortable with measurement contributes to the progress children make in the domain of calculating through estimating, while good insight into the meaning of measuring numbers also forms the basis for gaining insight into decimal numbers. Those internal connections between the various domains cause children to see mathematics more and more as a coherent whole.

**Structure of the learning-teaching trajectory for measurement**

**First intuitive measuring experiences**

The most elementary form of measuring, namely comparing, is often already performed by children naturally: comparing who has the bigger feet, who can build a higher tower, who has the longer piece of licorice, and so forth. In doing so, they find out that comparing can take place in various ways: by laying the objects next to each other, placing them on top of each other, by standing back-to-back with one another and such. In this way, a first, intuitive awareness of measuring emerges in many children. Also, they see adults around them being busy with all kinds of measuring. These are mostly advanced forms in which measuring instruments like the folding ruler, thermometer, scale or measuring jug are used. The fact that this involves measuring, is not totally clear to the children. For example, what exactly happens when they stand on the scale themselves, when their height is determined with a measuring stick, or when they see their mother weighing ingredients in a measuring jug, escapes them for the most part. But the fact that it means something is very clear to them.

They often feel strongly attracted to this kind of activities and will proceed to copy them.

Bente (just over two years old) takes a bath. While her mother undresses her, she sees the bath thermometer. She picks it up and announces that she is going to measure her feet. She then holds the thermometer along the side of her feet.
in a way she has seen lately, when adults try to determine her foot length. She moves her finger along the measure scale visible on the thermometer and concludes, “Six, mommy, it is six...”

The measure scale on the bath thermometer apparently brings out associations with the measuring stick she has seen on various other occasions. She realizes that measuring her feet is indeed an important activity, which has something to do with her own growth process and with the fact that shoes and other clothing items have to fit well. But how measuring works exactly is not completely obvious. And the fact that a bath thermometer does show some external resemblance with the measuring stick, but is used for very different purposes, probably eludes her completely.

**Basic pattern of the learning-teaching trajectory: from comparing through pacing off to reading off**

At the time children enter primary school, they usually already have had a number of the experiences like those described above. As a consequence, there is already a certain awareness of what measuring is and of what the various physical quantities are related to. The details of many measuring activities, however, often still elude them, and many new experiences in different fields have to be gained in school. In this process, it is important that the gap between their own meaningful measuring activities such as comparing and ordering on the one hand, and the imitated measuring activities with “adult” measuring instruments on the other hand, is gradually narrowed. How is this possible? Well, as was already concluded before, there is one crucial link that connects measuring through comparing with measuring through using a measuring instrument: measuring through pacing off a measurement unit like a step, a count, a cup, or a meter stick. In direct comparing, these measurement units are not yet used, just as taking measure in the sense of finding out how many times a given measurement unit fits in or alongside a certain object. When children have experienced the need for those measurement units and have gained the necessary experience with the use of them in all kinds of situations, the shift can take place in a natural way.
toward measuring with the help of measuring instruments, like those which they have gotten to know to a certain degree, in their own surroundings. During the latter, the measure is not taken by pacing off and counting anymore, but by reading off from a measure scale: for the number of times a measurement unit fits in or alongside a given object, a number is read off which coincides with the number of times that this measurement unit was paced off earlier. The actual pacing off is replaced by reading off, and the measurement activity is reduced to determining a height or distance on a measure scale. It is exactly when the children go through this process of measure development and shift toward the use of measuring instruments consciously, that insight into the essence of measuring is gained more and more deeply. As a result, a basic pattern for the learning-teaching trajectory presents itself, which is followed for the various physical quantities: from comparing through pacing off to reading off.

In the light of a series of activities concerning the theme of “cooking,” the children are intensely busy with baking cakes and such. Of course, ingredients have to be measured as well. Initially, the cup functions as a measurement unit in that process. For example, four cups of “flour” are measured, three cups of “milk,” two cups of “sugar” and so forth.

During the circle activity, the activities are once again reviewed. For example, a big transparent bottle is shown and the children get to estimate how many cups of “milk” can be gotten from a full bottle.

Through checking and putting marks on the bottle, the idea pops up of putting marks on the bottle per cup, in order to indicate how high the “milk” should come for one cup, two cups and so forth.
When numbers are added alongside the measure scale, a real “cups-measuring-jug” is created, which may prove valuable in measuring ingredients.

**Central position of the physical quantity length**

Within the learning-teaching trajectory for measurement through which children move in the lower grades of primary school, the physical quantity length takes a central position. For this physical quantity, the phasing described above, of:

- measuring through comparing and ordering
- measuring through pacing off using a measurement unit
- measuring through reading off with the help of a measuring instrument

is treated in its entirety. For example, in kindergarten 1 and 2, extensive attention is given to comparing and ordering, but also to pacing off a measurement unit, during which body measures play an important role. Also, a first orientation takes place on the use of measuring instruments in the form of the “five-meter-tape” (retractable measuring tape). An orientation which is continued and elaborated on in grades 1 and 2, when the children construct a “footmeter” for the first time, and afterward investigate more conventional measuring instruments like the folding ruler, household measuring tape and board ruler, and learn how to use them. Because of the fact that the measure scale of these instruments has a strong resemblance with that of the footmeter they have constructed themselves, insight into the how and why of these instruments continues to grow. In many of these activities involving the physical quantity length, measuring by themselves receives a lot of attention. Their own, practical experiences in this field are important. Moreover, mainly the questions aimed at further reflection and shortening of the operation of measuring make measuring activities valuable:

- what would be a suitable measuring strategy?
- how do I perform this measuring, practically speaking?
- how do I interpret the result of the measuring?
- could I do it in a handier and quicker way?

The collective answering of questions such as these leads to a growing skill in practical measuring and to a growing elaboration in the work methods that are used in the measuring of this physical quantity.
In grade 2, the children first gain more experience with measuring longer distances and sizes with the help of a meterstick: a stick with a length of exactly one meter. This leads to measurement results like “almost four meters wide” and “two meters and a bit more high.” Next, the teacher tells a story about a land in which a person’s foot is used as a measurement unit. In that country, people use a measure stick in the length of one standard “foot” as a measure to perform all kinds of physical quantities through pacing off. But in the long run, it turns out that this is not really handy. Especially when there is a lot of measuring to be done (like the height of walls under construction), pacing off is very time-consuming. More and more people are starting to wonder, “Could it not be done in a faster and simpler way?”

The children receive a “footstrip,” a long sheet of paper and a pair of scissors, and try to think of a solution for this problem in their group. Soon, the idea occurs in various groups to make a “footmeter”: you take the long sheet, mark eight or ten “feet” on it, and then you only have to put the strip alongside the object you are measuring and count the number of “feet.” Other groups go a bit further. They argue that you can also write numbers next to the marks of the “feet,” and then you have a real footmeter from which you can read off the length.
Other physical quantities: exploration in connection with length

For the physical quantities of volume, weight, area and time, the phasing is gone through in a less elaborate and complete way. Still, activities in these fields are of great importance. Every physical quantity does have its own specific possibilities and difficulties when it comes to comparing objects and taking measures. It is exactly because the children gain the needed experience with each, that they obtain a better insight into the essence of measuring, while they also develop a certain level of practical skill in measuring. What it mainly comes down to now, is that activities involving these physical quantities are chosen in a way that:

- they add to and enrich what the children have already experienced with length in these different phases
- the connection between the different physical quantities regarding the work methods used in measuring are brought forward more directly.

In this way many valuable activities take place in kindergarten 1 and 2 in the field of volume and weight, in which comparison strategies are developed that provide an expansion of what was already discovered on this subject in length. For volume, the emphasis in these classes is put on pacing off with a measure and on the use of a simple measuring instrument in the form of the cups-measuring-jug described above. The importance of this is, of course, not so much the instrument’s practical value, but more in consciously experiencing the shift from measuring through pacing off to measuring through reading off, and the fact that the measuring in this process is transformed, as it were, into a form of length measuring, during which the measure is read off of a linear measure scale.

Also in the field of measurement of area, many valuable activities involving comparing and pacing off using a measure are possible, but these are more suitable for grades 1 and 2. In a surprising way, it also becomes clear how one physical quantity can sometimes be replaced by another one in order to make the measurement more feasible.

In grade 1, the children have gained the necessary experiences during the past weeks with comparing the areas of various flat objects like pieces of cloth and small gardens. These were mostly
activities on paper, in which, using pictures of the objects in question, appropriate comparison strategies were sought, like placing objects on top of each other and checking what sticks out, cutting out, reshaping, and so forth. Today, all this is done again: three pieces of “baking dough” (clay), of almost equal thickness, are in the middle of the circle. The question is: which piece is the biggest? And what would be a clever way to find out? Several strategies are brought forward that some of the children have already tried. Additionally, it turns out there is another surprising new suggestion: you can put the pieces on your hand and try to feel which one is the heaviest. And if that does not work directly, you can put them on a scale and see which one is the heaviest.

For the physical quantity weight, children in grades 1 and 2 also experience how it connects with measuring of length when constructing a spring balance themselves. Previously, they have worked with a scale with a set of standard weights of 10, 50, 100 and 500 grams. While doing this, they discover how these standard weights are used to construct an instrument with which the weight can be read off of a measure scale. In this way, they experience again how measuring through pacing off can shift into a form of measuring through reading off through a transformation into the physical quantity length. Here too, the principle applies that the exploratory activities with these measuring instruments provide important broadening and enrichment to what has already been experienced earlier about the physical quantity length. Moreover, insight into the connection between the various forms of measuring grows. The value of working with those measuring instruments themselves lies not only in the fact that insight in measurement is strongly improved. In the measure scales that are characteristic for these instruments, a part of the measure systems is already included, which will be the subject of study and reflection more and more from grade 3 onward. For example, on the chalkboard ruler, the system of small length measures is included – from meter down to the millimeter – and the system of the weight measures of kilogram, gram, ounce and pound is included on the kitchen scale. As a result, through working with measuring instruments, one anticipates the exploration of the various measure systems as such, and, in connection with that, calculating and reasoning with measures.
The goal and organization of measurement education

Emerging measure sense as an overarching goal

The goal of measurement education is to teach the children to measure with an appropriate measure, to use appropriate mathematical language and to solve simple measurement problems with length, circumference, area, volume, weight, speed and temperature.\textsuperscript{5}

In kindergarten 1 through grade 2 of primary school, the foundation must be laid for the achievement of this goal. As was described before, the emphasis in this lies on gaining a number of meaningful measuring experiences, through which a connection is made between the own, informal measure knowledge and the use of conventional, but not yet very well understood measuring instruments like the ruler, scale and measuring jug. Knowledge of the different kinds of measures, both for natural measures and standard measures, gradually starts to develop. Also, the children progressively gain more insight into the working of the specified instruments, especially regarding the meaning of reading off measures on a measure scale.

Through the intermediate attainment targets formulated in the following chapters, landmarks are set out that can help the teacher to realize measurement education. However, there is also an overarching goal which forms the basis for the more specific goals. In measurement education, the first and foremost goal is that the children develop a sense for measuring by:

- developing awareness of the type of situations in which measuring can be used as an approach
- learning how to approach phenomena and situations in everyday life quantitatively, in a fitting manner
- developing an appropriate measurement language
- improving the ability to distinguish the different physical quantities and determine which measurement is appropriate in which situation
- learning to imagine the various kinds of measures that are connected with the physical quantities.

In other words: the activities and experiences should, in the first place, contribute to the children’s emerging measure sense.
If this is the case, the more specific goals in the domain of measurement can be achieved without much trouble. That is the reason why, in the following chapters, references will be made regularly to this overarching goal of measurement education.

Organizing measurement education
Educational activities in the domain of measurement are sometimes accompanied by complications resulting from working with all kinds of separate materials. This mainly concerns the objects to be measured, as well as the devices and measuring instruments that are used. Working with these materials can create a messy environment in the classroom, in which it is not always easy for teachers to maintain good oversight and make sure everybody stays involved in the activities. One result of this is that teachers (and textbook authors) are sometimes tempted to pass on the real measuring activities and to limit themselves to a minimal introduction of the main standard measures, followed by paper activities, in which the main purpose is to shift from one type of measure into another. How many centimeters are there in a meter? How many grams in a kilogram? And so forth.
Obviously, it is quite understandable that teachers want to remain in control of the classroom, and will, in many cases, choose a simple approach to activities. On the other hand, partly or totally skipping real measuring activities raises strong objections, because it diminishes the essence of the learning experiences that students can obtain through measuring. Perhaps even more strongly than in other domains of mathematics education, the principle applies that the child’s own discoveries and the children’s self-gained insights are of extreme value. Thinking up handy comparison strategies by themselves, investigating together how to measure through pacing off with a meterstick, and reflecting on the hows and why's of measuring instruments with the entire class, for example with the measuring jug, the ruler and the letter scale—this is still what it comes down to primarily. This is the foundation that gives sense and meaning to the later shifting from one measure type to another and to calculating and reasoning with measures. A firm and coherent organization of education is essential in this process. Gathering the needed materials beforehand and reflecting on the activity can be extremely helpful in this.

In this context, one can answer questions like:
- Exactly which materials are needed?
- In which amounts should these be present, in order for everybody to remain active?
- Would it be preferable to perform the measuring activities with the entire class, or is it better to have the children working in groups?
- In the latter case: is it preferable to have a kind of task division, in which every child knows what his/her task is?
- In what way do the children write down the results of their measurements?
- What should children do when they finish their measurements early?
- How might a follow-up, whole-class discussion take place?
Of course, it is impossible to organize extensive measuring activities on a regular basis. It would require so much preparation and organization that the other lessons could possibly suffer from it. But organizing these activities from time to time is essential for proper measurement education. One measuring activity, actually carried out and collectively discussed, can induce more than ten measuring problems on paper. And the children’s enthusiasm is often so great, that a certain level of loud busyness will be lovingly accepted.

A few points viewed more closely

**Historic excursion: money as a physical quantity**

To what extent should money be regarded as a separate physical quantity? And to what extent does this subject deserve a separate place in the learning-teaching trajectory? Opinions may differ on this subject. For example, money is considered a part of measurement in some of the current mathematics textbook series, while it is viewed as part of the domain of numbers and number relations in other textbooks. In the latter case, it is regarded mainly as a cornerstone for number concept, as well as an important topic in its own right.

Historically, it would certainly be defensible to regard money as a separate physical quantity in the sense of value. From the time that trade between different people in the ancient past started developing, this physical quantity has played a role in the form of trade value. Initially, this involved barter, in which steady exchange relations were mostly used. When trade became more intensive and far-reaching, the need increasingly grew for specific valuable means of exchange: resources that were not specifically foods, but which independently represented an intrinsic, steady high value. Precious metals like gold and silver were especially well-suited for this purpose. While value as a physical quantity up to that point had mostly been connected to number and volume, the relationship with weight became stronger from that point on.
By using gold rings or silver bars of a certain weight, for example, it became possible to trade in a more advanced way: for example, people traded pigs or sheep for a certain number of gold nuggets or silver bars, and then used these to buy grain or salt.
The silver bars obviously needed to have a constant weight, and this was obtained by linking its weight to the weight of a number of grains of corn. The balance functioned as a helpful instrument for calibration. In order to avoid tampering with these weights, from a certain point in time, the bars were cast usually in a round shape, and stamped with the picture of a monarch. With this, money as we still use it today was introduced: a system of coins and bank notes that represent a constant value and that can be used in the purchase and sale of all kinds of goods. The fact that this development is nowhere near its completion can be seen from the flux of new electronic forms of payment: PIN card, credit card, internet banking and so forth. In this process, money is limited more and more to being a calculating unit in the payment circuit, and its role as a direct form of payment is becoming less and less prominent.

Of course, it is important that children become thoroughly acquainted with our monetary system, especially in a time when we are shifting from one monetary unit to another: from the guilder to the euro. However, it is not necessary that this is done by regarding money as a physical quantity that in school should be treated in the same way as other physical quantities, like length or weight. In the first place, this is due to the fact that money involves a so-called discreet physical quantity, as opposed to the earlier-mentioned continuous physical quantities like length and volume. Specifically in money, not all values can be automatically expressed in a number of monetary units. In connection with that, the real measuring activities, in the sense of comparing values or determining value, are not really possible in this physical quantity. Also, there are no real measuring instruments available for determining its value. At best, you could consider the digital scale, which is used in supermarkets to weigh and price apples or tomatoes, somewhat as such. Activities regarding money generally do not contribute much to the overarching goal of emerging measure sense. For that reason, no room is made in the next chapters for a separate learning-teaching trajectory regarding the physical quantity money.
A special position for the physical quantity time
The physical quantity time has a special position within the learning-teaching trajectory. Here, too, natural activities are possible in which the already mentioned three types of measuring can play a role. For example, time can be measured or marked out through counting and through the use of instruments like the hourglass, egg timer, stopwatch and the (digital) watch. In that process, important measurement units, like minute and second, can be further explored and experienced. But the emphasis on this physical quantity is still mainly on a different aspect, namely the development of a good sense of time and – in connection with this – on learning to tell time. The development of time sense plays a role throughout primary school, while learning to tell time, regarding analogous time indications, is mainly emphasized in grades 1, 2 and 3. Working with digital time indications is mainly treated in grades 3 and 4, during which the connection with “regular” time telling is an important point of departure. While learning to calculate and reason with (digital) time is a topic mainly in the higher school grades, for example in working with arrival and departure times. In that sense, this physical quantity plays a different, yet still quite important role in the following learning-teaching trajectory description.

Graphs in measurement education
In practice, measurement results are often presented in a variety of ways. One of those ways is through graphs. Through the use of various forms of representation or various types of graphs, it is possible to present a large number of measurement data in a well-organized manner. Moreover, certain connections can be discovered and conclusions made through the use of a graph. As a result, graphs form a topic that certainly deserves a place in education regarding measurement. The emphasis is not so much on reading and analyzing existing graphs, but mostly on constructing graphs themselves, based on self-made measurement data.
In the following learning-teaching trajectory description it is illustrated how this graphical depiction of measurement results can begin to take place in an early phase of primary school. One example is when children make a strip of their own height, and then hang these strips next to each other on the wall. Afterward, similar bar graphs are created for the children’s weights and the size of their waists. Also, the children experience how the passing of time can be expressed graphically in various ways, in the form of daily, weekly and monthly schedules. For the time being, this will be limited to simple pictograms, bar graphs and pie charts.