THE INTERNATIONAL HANDBOOK OF MATHEMATICS TEACHER EDUCATION

VOLUME 2

Tools and Processes in Mathematics Teacher Education

Dina Tirosh and Terry Wood (Eds.)

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TOOLS AND PROCESSES IN MATHEMATICS TEACHER EDUCATION

The International Handbook of Mathematics Teacher Education

Series Editor:

Terry Wood
Purdue University
West Lafayette
USA

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Edited by

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and

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SENSE PUBLISHERS
ROTTERDAM / TAIPEI

A C.I.P. record for this book is available from the Library of Congress.

ISBN 978-90-8790-544-6 (paperback)
ISBN 978-90-8790-545-3 (hardback)
ISBN 978-90-8790-546-0 (e-book)

Published by: Sense Publishers,
P.O. Box 21858, 3001 AW Rotterdam, The Netherlands
http://www.sensepublishers.com

Cover picture:

The rich flowers could be seen as the profusion of rich tasks, attracting students to engage, and which belong in a highly complex space of intertwining branches. Only when students engage in several related tasks is there a chance of pollination, for the transformation of activity into understanding.

Tasks that seem similar may be ‘far apart’ mathematically, tasks which seem far apart may be ‘close’ mathematically. The real effect of engagement in tasks is the ongoing life of the tree, and the production of ‘seeds’ which require the transformational effects of photosynthesis and sap (reflection and integration) in order to mature.

The tree can be seen as the teacher’s knowledge, firmly planted in the soil of mathematics, and flowering profusely, a reminder of the variety of teacher proficiency required to teach effectively and indicating the mathematical connections and structures of which they are aware. The intertwining vine is a reminder of the necessity of depth as well as breadth of connections in order to constitute understanding.

John Mason 2008

Printed on acid-free paper

This book is dedicated to the memory of
my dear father
who always believed in me
and encouraged me

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PREFACE

It is my honor to introduce the first *International Handbook of Mathematics Teacher Education* to the mathematics education community and to the field of teacher education in general. For those of us who over the years have worked to establish mathematics teacher education as an important and legitimate area of research and scholarship, the publication of this handbook provides a sense of success and a source of pride. Historically, this process began in 1987 when Barbara Jaworski initiated and maintained the first Working Group on mathematics teacher education at PME. After the Working Group meeting in 1994, Barbara, Sandy Dawson and I initiated the book, *Mathematics Teacher Education: Critical International Perspectives,* which was a compilation of the work accomplished by this Working Group. Following this, Peter de Liefde who, while at Kluwer Academic Publishers, proposed and advocated for the *Journal of Mathematics Teacher Education* and in 1998 the first issue of the journal was printed with Thomas Cooney as editor of the journal who set the tone for quality of manuscripts published. From these events, mathematics teacher education flourished and evolved as an important area for investigation as evidenced by the extension of JMTE from four to six issues per year in 2005 and the recent 15th ICMI Study, *The professional education and development of teachers of mathematics*. In preparing this handbook it was a great pleasure to work with the four volume editors, Peter Sullivan, Dina Tirosh, Konrad Krainer and Barbara Jaworski and all of the authors of the various chapters found throughout the handbook.

In Volume 2, *Tools and Processes in Mathematics Teacher Education***,** edited by Dina Tirosh, various promising tools and processes that are aimed at facilitating the mathematics teacher development are described and critically analyzed. This volume provides a look at how mathematics teacher educators think about and approach their work with teachers. This is the second volume in the handbook.

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West Lafayette, IN
USA

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DINA TIROSH

TOOLS AND PROCESSES IN MATHEMATICS TEACHER EDUCATION

An Introduction

The title of this volume includes five concepts that carry many definitions and interpretations: *tools, processes, mathematics, teacher, education*. Of these, two are mainly addressed in this (second) volume of the International Handbook of Mathematics Teacher Education. The first is "tool" and the second is "process".

The use of tools is viewed as an important step in the evolution of mankind. The production and usage of the first tools, that were probably made of stone, started at the Stone Age (and possibly even before), and provided our ancestors with a mechanical advantage that was not naturally available to them. The nature and spectrum of tools have vastly expended during the years to include physical, conceptual and symbolic tools (language, for instance, is often viewed as the most important symbolic tool provided by society). Nowadays, we live in a world that is increasingly dependent on tools.

In this volume, a range of tools that are often used in mathematics teacher education, all aimed at facilitating various proficiencies needed for teaching mathematics, are described and critically analysed. Obviously the choice of a tool to be used for a certain aim is a major decision taken by a mathematics teacher educator. Yet, the process (the particular manner in which the tool is used) is not less important. The effectiveness of a tool is largely determined by the specific manner in which it is implemented (e.g., the role of the mathematics teacher educator, the roles of the prospective and/or practising teachers, and the nature of the tasks). Awareness of the scope of a tool and its limitations is essential for wisely using it.

Getting to know the tools and the processes is a demanding, time-consuming endeavour. The authors of the chapters share with us their invaluable experience in employing the tools in mathematics teacher education. This accumulated experience could assist us in making decisions about both the tool(s) and the process(es) to be used for various purposes in mathematics teacher education.

In this introduction I first briefly summarize the major themes of each chapter according to their order of appearance in the volume. Then, I suggest some central questions to consider while reading the chapters.

D. Tirosh and T. Wood (eds.), Tools and Processes in Mathematics Teacher Education, 1–11.

SECTION 1: CASES IN MATHEMATICS TEACHER EDUCATION

Cases in mathematics teacher education vary in many aspects, including the intended audience (e.g., prospective elementary school teachers, prospective secondary school teachers, practising elementary school teachers, practising secondary school teachers), the mathematical content (e.g., specific mathematical ideas, general mathematics structures), the pedagogical content (e.g., students' reasoning and ways of thinking, socio-mathematical norms of explanations and justifications), the type of media (e.g., narrative, video), authorship (e.g., teachers' own experiences and practices, written by a third party) and authenticity (e.g., a depiction of real events, "invented" situations). In this volume, four types of cases that are often used in mathematics teacher education are described and discussed. These are narratives, mathematics case discussions, video-recordings, and lesson studies.

In the first chapter, Olive Chapmann describes and discusses the use of narratives in mathematics teacher education. For Chapmann, narrative is a key form through which individuals come to know themselves, construct their lives, and make sense of their experiences. In her chapter, narratives refer to self-authored, written or oral stories, which allow teachers to present their teaching or learning experiences from their perspectives and to weave descriptions of their thinking, feelings, attitudes, and other personal attributes relevant to these experiences into the stories. Her review of the literature indicates that, beginning in the mid-1990s, there was a small but growing body of work involving the use of narrative in mathematics teacher education.

Chapmann describes the use of narrative as a research tool for studying teacher knowledge and teaching, and as a pedagogical tool for teacher professional development and education in prospective and practising mathematics teacher education, and then focuses mainly on the use of narratives as a pedagogical tool with prospective teachers in mathematics education. She organises her description of her ways of using narratives in two themes: writing the narratives and narrative reflection/analysis/inquiry. The section on writing the narratives provides practical suggestions of types of guidelines that are likely to elicit stories that address mathematics pedagogy as opposed to generic pedagogy, along with useful examples of guidelines and of stories written by prospective teachers. The section on narrative reflection/analysis/inquiry presents eight ways in which prospective secondary teachers can be engaged in learning through narratives (e.g., narrative inquiry through peers, comparing imagined versus actual actions, and comparing personal versus theoretical knowledge). At the end of the chapter, Chapmann raises two, broad issues for future research: What is the role of narrative in the construction of mathematics teachers' knowledge? What is the meaning and usefulness of narrative to teachers learning to teach, as well as to experienced teachers? She concludes by stating that there is still a lot that could be done with narrative as a pedagogical tool and even more to understand about it as a process and product of research.

While Chapman's chapter (Chapter 1) addresses self-authored stories, Zvia Markovitz and Margaret Smith (Chapter 2) focus on cases that are written by a third party. They describe two broad classes of cases: exemplars and problem situations. Exemplars are lengthy narratives that portray an instructional episode in its entirety, highlighting the actions and interactions of a teacher and her students. Problem situations are short scenarios that focus on specific problems or dilemmas encountered by teachers as they listen to and interact with students. The main body of the chapter provides in-depth examples for each of these two classes of cases. The example of an exemplar is based on the work of Smith and her colleagues in the United States, and the examples of problem situations on the work of Markovitz and her colleagues in Israel. The chapter clearly specifies the characteristics of each of these types of cases and provides valuable suggestions for engaging prospective and practising teachers in meaningful activities related to a case (e.g., encourage the teachers to solve and discuss the mathematical task on which the case is based and to generalise beyond the case). Markovitz and Smith, much like Chapmann, conclude the chapter with several, central issues in need of further study: What do teachers learn from different types of cases? How do they learn it? How what teachers learn impacts their teaching performance?

Carolyn Maher (Chapter 3) opens a window on using videos as pedagogical tools in mathematics teacher education. Maher demonstrates that (and how) videos can provide prospective and practising teachers with opportunities to closely observe students' emerging processes of learning, analyse teaching episodes in a variety of classroom conditions for learning and teaching, and study their own teaching. She argues that in comparison with other evidence of students' learning, such as verbal statements and explanations, written work, and other forms of evaluation, video recordings can capture the moves that are made and thus offer the opportunity for studying subtle details of students' evolving learning.

Maher reviews various projects that use video collections as tools in mathematics teacher education, including video cases (e.g., the STELLAR system, University of Michigan 'Library Collection' and the Private Universe project in mathematics) and video recordings that were initially developed for research processes beyond teacher self-study (e.g., TIMSS and the Longitudinal Study of Student learning at Rutgers University). She then provides detailed examples of how video collections have been used as pedagogical tools for the study of learning and teaching and for mathematics teacher education. The example of the use of the longitudinal study of student learning at Rutgers university, initially developed for tracing the mathematical thinking and reasoning of students from a low Socio-economic (SES) community, can serve as a model for a potential, fruitful interplay between using videos as a research tool and as a pedagogical tool.

The first three chapters in this section presented three types of cases that are described as relatively new tools in mathematics teacher education. Makoto Yoshida (Chapter 4) introduces the *lesson study*, a professional learning process that originated and has a long history in Japan, and is gaining increased attention in the United States and in a growing number of countries in Asia and Europe. Lesson study is a process of professional learning that practising Japanese teachers engage in continuously throughout their teaching careers to examine systematically their instructional methods, teaching content and curriculum, as well as their students' processes of learning and understanding. Yoshida discusses, in his chapter, three main forms of lesson study that are used for various purposes in Japan: school-based lesson study, district-wide lesson study, and cross-district lesson study.

Yoshida describes the major, three activities that constitute the lesson study: establishing a lesson study goal; engaging in a lesson study cycle to develop, implement, observe, and reflect on research lessons; and reflecting on the whole lesson study process by producing a written report. He then examines, in detail, one of the important parts of the lesson study process, *kyozaikenkyu* (instructional material investigation) and illustrates, in a vivid manner, the various features of this inquiry, using an example of a research plan on the area of triangles. Towards the end of the chapter, Yoshida raises and provides in-depth answers to two major questions: Why lesson study? What does it provide? The chapter concludes with valuable ideas for conducting effective lesson studies.

The four chapters in this section provide a glimpse into various types of cases that are used in mathematics teacher education. Readers are advised to continue thinking about the similarities and differences between these and other types of cases that are commonly used in mathematics teacher education and about the relevance of each of these types of cases for different aims in prospective and practising mathematics education.

SECTION 2: TASKS IN MATHEMATICS TEACHER EDUCATION

This section presents several, predominant tools that are used in mathematics teacher education. Two chapters (Chapter 5 and Chapter 6) describe textual tools (written tasks and examples). The other two chapters (Chapter 7 and Chapter 8) focus on physical tools (manipulatives and machines). The major issue that is addressed in each of these four chapters is: Which tasks have been significant for the learning and development of mathematics teachers?

Anne Watson and Peter Sullivan (Chapter 5) clarify that the centre of attention in their chapter are classroom tasks. They suggest several ways in which the study of classroom tasks and mathematics lessons can be used to enhance teacher learning. They note that mathematics teacher educators can use the obvious interest of prospective and practising teachers in planning and teaching lessons to draw their attention to major aspects of student learning. They then identify several purposes for using classroom tasks and mathematics lessons in mathematics teacher education, including highlighting the range and purpose of possible classroom tasks, stimulating teachers' theorising about students' learning, providing opportunities to learn more about mathematics and about the nature of mathematical activity.

The four, major parts in Watson and Sullivan's chapter are devoted to describing and discussing four types of tasks, each of which aimed at fostering one of four of the major strands of mathematical proficiently (as suggested by

Kilpatrick, Swafford and Findell, 2001, with some adaptations): conceptual understanding, mathematical fluency; strategic competence, and adaptive reasoning. In each of these parts they elaborate on the nature of the learning associated with the strand, offer illustrating tasks that mathematics teacher educators can use with the teachers and propose a template of a lesson structure to which tasks within that strand can be adapted.

Rina Zazkis (Chapter 6), much like Watson and Sullivan, starts by identifying the two major goals at which the specific types of examples that she describes and discusses in her chapter are aimed, namely, enhancing teachers' personal understanding of mathematics, and introducing the variety of students' possible understandings or misunderstandings of mathematics. She focuses on teachers as learners and suggests instructional examples that are effective in increasing teachers' awareness of basic assumptions that are parts of information used in mathematical activity, but not mentioned explicitly in the tasks. Zazkis shows how examples that are aimed at challenging basic assumptions related to mathematical content can be used as effective tools to increase teachers' mathematical understanding and to invite them to re-examine their very basic assumptions related to the teaching and learning of mathematics.

Zazkis distinguishes between three kinds of basic assumptions: mathematical conventions, shared understandings, and assumptions that present unintended constraints to problem solving. She then provides several, illustrative examples that she often uses for challenging each of these three kinds of basic assumptions and discusses their effectiveness in increasing teachers' mathematical understanding and pedagogical sensitivity. Zazkis concludes by arguing that the presented examples not only extend the teachers' example spaces, but also act, through creating dis-equilibrium, as significant catalysts for the construction of richer cognitive schemas that are relevant to the learning and teaching of mathematics.

Marcus Nührenbörger and Heinz Steinbring (Chapter 7) propose, in their chapter, that manipulatives have a dual role in mathematics teacher education. On the one hand, they serve as learning objects for prospective teachers, in the sense that various manipulativies can be used to promote their own learning and understanding of mathematics. On the other hand, prospective teachers need to gain practical knowledge on the use of manipulatives in mathematics instruction. For example, they should learn which manipulatives are often used in various grade levels, how to appropriately employ them, and how to analyse and diagnose the students' learning processes when using manipulatives.

Nührenbörger and Steinbring introduce a theoretical perspective on the role of manipuatives in mathematics teacher education and provide several examples to illustrate the various aspects of this perspective. A major argument is that manipulatives and mathematics are initially separated from each other and a relation between them can only be productively constructed by the learner's own interpretations of these relationships. Thus, the value of a specific manipulative in both mathematics education and mathematics teacher education depends on the relationship between this object and the mathematical relations and structures it represents. Nührenbörger and Steinbring observe that prospective teachers have their own, often naive views of the nature, role and effects of manipulatives on mathematics learning. They argue that it is necessary to confront prospective teachers, in their education, with essential theoretical aspects related to the use of manipulatives, as well as with concrete material. Such a confrontation is apt to cause cognitive conflict and thus to encourage changes in existing subjective conceptions and to stimulate further developments. In their concluding remarks, Nührenbörger and Steinbring suggest several guidelines for evaluating manipulatives (e.g., in what way do the manipulatives embody basic, structured, mathematical ideas? Is it possible to continue with the manipulatives using the same structure in different school years? Are the manipulatives well structured and manageable for students?). They clarify that knowledge about interpretations and use of manipulatives evolves through interactive learning and not in a procedural manner.

Maria Bussi and Michela Maschietto (Chapter 8), in the last chapter in this section, invite the readers to “pay a visit” to the Laboratory of Mathematical Machines of the University of Modena and Reggio Emilia, a well known research centre for the teaching and learning of mathematics by means of instruments. This laboratory holds a collection containing more than 200 working reconstructions (based on the original sources) of many mathematical instruments taken from the history of geometry. The laboratory also contains arithmetical machines and Information and Communication Technologies (ICT). In this chapter Bussi and Maschietto focus on the use of concretely handled machines that require motor abilities. They argue that it is a true challenge for teachers to effectively use such artefacts in their mathematics classrooms – it requires specific professional competences, which cannot be taken for granted.

In the two major parts of this chapter, Bussi and Maschietto describe and discuss two examples, the first addresses the use of arithmetical machines related to place value in primary school and the second focuses on using geometrical machines related to symmetry in secondary school. In each of these two parts, Bussi and Maschietto first define the mathematical object that is mediated and describe the related network of artefacts (counting sticks, spike abacus and pascaline in the first part, geodreieck and symmetry linkage in the second). They then describe, in brief, the mathematical meanings that could potentially be attached to these artefacts, analyse the related *instrumentalisation* (the emergence and the evolution of the different components of the artifact, drawing on the progressive recognition of its potentialities and constraints) and *instrumentation* (the emergence and development of the utilization schemes) processes, and offer carefully chosen, related tasks to be use in teacher education. Bussi and Maschietto conclude by noting that the use of physical manipulations is becoming less frequent in mathematics education and that they are too often substituted by ICT. They argue that the features of these two systems (physical manipulations, ICT) differ and that each system should have its own place both in mathematics classroom and in teacher education.

Before ending the brief description of this section, I would like to emphasize the obvious: Only four main types of tasks are described and discussed here. There are, however, many other types of tasks that are used in mathematics teacher education. An issue deserving attention is: What other tasks in mathematics teacher education have been significant for the learning and development of mathematics teachers?

SECTION 3: RESEARCH AS TOOLS IN MATHEMATICS TEACHER EDUCATION

The third section suggests several ways in which research in mathematics teacher education contributes to the development of tools and processes in mathematics teacher education. The chapters in this section focus, in various ways, on the accumulated research on common students' conceptions and habits of thinking and offer ways of using this knowledge as tools in mathematics teacher education.

Pessia Tsamir (Chapter 9) raises, in her chapter, three major questions: What theories should be presented in mathematics teacher education programs? In what ways and at what stages (e.g., prospective teachers or practicing teachers) should theories be presented? How to assess the impact of teachers' familiarity with theories on their professionalism? She clarifies that the purpose of her chapter is to examine whether theories can serve as tools to promote prospective teachers' and practicing teachers' (1) mathematical knowledge (an SMK component), (2) knowledge about students' mathematical reasoning (a PCK component), and (3) knowledge about the formulation and sequencing of tasks (a PCK component).

Tasmir focuses on two theoretical models that she discusses with secondary school prospective and practising teachers in her courses: The three knowledge components (intuitive, algorithmic, formal) theory and the intuitive rules theory. She then presents three research segments to illustrate that familiarity with these theories can promote prospective and practising teachers' knowledge (mathematical and pedagogical) and teaching practices. She concludes by noting that the ultimate aim of mathematics teacher education is to provide teachers of mathematics with the professional learning opportunities they need to lead their students to succeed in learning mathematics. Thus, she argues, an evaluation of the impact of teachers' familiarity with specific mathematics education theories should not be restricted to evaluating changes in their SMK, PCK and teaching practices. Tsamir urges the mathematics teacher education community to devote more efforts to study the outcomes of teachers' familiarity with theories on students' knowledge and mathematical confidence.

The body of knowledge on young children conceptions, preconceptions, alternative conceptions and misconceptions of number, measurement and geometry is notably wide. In view of this volume of knowledge, theoretical frameworks covering categorization, general sources, and appropriate educational approaches seem to be in order. Such frameworks, if presented in a teacher-friendly manner, have a potential of being a viable tool in mathematics education. Barbara Clarke outlines, in her chapter (Chapter 10) how she and her colleagues developed frameworks of research-based growth points in number, measurement and geometry. She then described how these frameworks, together with a closely-related one-to-one, task-based assessment interview, were used in a professional development program for grade K-2 practising teachers in the Early Numeracy

Research Project (ENRP) in Australia. A main aim of the project was that the practising teachers would, in time, carry the frameworks in their heads, using it as a kind of “lens” through which they could view interactions with children individually, in small group or whole class interactions, as well as during lesson planning.

Clarke focuses, in her chapter, on the impact of the growth points on practising teachers. In several questionnaires, teachers were invited to identify the aspects of their teaching practice that had changed as a result of their involvement in the ENRP. The most common theme emerging was summarised as “growth points inform planning,” indicating the value that teachers saw in acquiring systematic knowledge of how children learn mathematics and how mathematical understanding develops. Some emotional, significant aspects were: having more confidence in teaching mathematics and enjoying mathematics more. These expressions of both professional and emotional changes should be viewed in light of the impressive impact of the project on the mathematical learning of the practising teachers’ students: Students in trial schools outperformed significantly children in reference schools, at every grade level and in every mathematical domain.

The first two chapters in this section mainly address the body of knowledge on children’s conceptions. Susan Empson and Victoria Jacobs (Chapter 11) focus on another aspect of teachers’ knowledge of students ways of thinking, that is, on knowing to listen, in a responsive manner, to the mathematical thinking that children use to solve problems, create representations, and make arguments. They cite research indicating that responsive listening to children’s thinking during instruction have multiple benefits including (a) improving children’s understandings, (b) providing a means of formative assessment, (c) increasing teachers’ mathematical knowledge, and (d) supporting teachers’ engagement in generative learning.

Empson and Jacobs provide an example of a teacher listening to a young child, highlighting four features of the interaction that are characteristic of responsive listening, including supporting the girl’s making sense of the problem without telling her how to solve it, asking her to articulate how she had arrived at her answer after she solved the problem, and maintaining a conversational interaction pattern instead of the traditional initiation-response-evaluation pattern. They then characterise three types of teachers’ listening: Directive, Observational, and Responsive, and detail three types of learning experiences that are instrumental for developing responsive listening: (a) discussions of children’s written work, (b) discussions of videotaped interactions with children, and (c) opportunities for teachers to interact with children and then to reflect on those experiences with other teachers.

The last chapter in this section largely departs from the other three. Koeno Gravemeijer (Chapter 12) focuses on *one* theory, the domain-specific instruction theory for Realistic Mathematics Education (RME). In this chapter he outlines the complexities of enacting instruction in tune with RME, describes three major, related competencies, and concludes by suggesting several principles for preparing prospective teachers for mathematics education that is in line with RME.

Gravemeijer exemplifies, in the first parts of the chapter, the complexity of designing instructional sequences in line with the RME theory. In view of that complexity he argues that there is no expectation from teachers to design such instructional sequences themselves, and notes that such instructional sequences are developed at the Freudenthal Institute and in other centres. Next, he specifies three major categories of roles and competencies needed for teaching in line with the RME theory, the first concerns planning and designing instructional activities and hypothetical learning trajectories, the second addresses the classroom culture, and the third focuses on the orchestration of the collective reinvention process. The golden rule for fostering RME teacher competencies in mathematics teacher education is to teach what you preach, a rule that evidently applies to this theory and to many other principles and theories.

The four chapters in this section address only few of the theories that are currently used in mathematics education. I encourage the reader to think about the potential contribution of theories that are not mentioned here to the development of tools and processes in mathematics teacher education.

SECTION 4: GENERAL PERSPECTIVES AND CRITICAL RESPONSE

Each of the preceding 12 chapters in this volume presents some tools and processes that are aimed at enhancing the knowledge-base and the skills needed to teach mathematics. The two chapters in this section take a different stance. Shlomo Vinner (Chapter 13) raises a very substantial, general issue: How to use the teaching of mathematics as a tool to promote general educational values? Vinner claims that some dimensions relating to this direction are missing from teacher education practice, including specifying the goals of teaching mathematics, defining the goals of education in general, identifying ways in which the goals of teaching mathematics can be used to achieve the general goals of education, and determining the meaning of the teacher's work.

Vinner demonstrates how some of the characteristics of mathematics can be used to promote general, educational values including the importance of procedures in society and in everyday life, analytical thinking, reflection and rational thinking. He then clarifies that both students and teachers feel better if they realize that they are involved in meaningful activities and points out at three contexts in which this notion of meaning is used: the meaning of notions, the context of values, and the meaning of life. Vinner claims that the latter context in which meaning is used is absolutely missing from school practices. He suggests that issues such as what is the meaning of our life and what are we supposed to do with our life, bother all adolescents, and therefore they should be discussed in mathematics teacher education. He clarifies that mathematics teachers are educators and as such they should be prepared to relate to all problems of their students and not only to the mathematical problems.

In the concluding chapter (Chapter 14), Alan Schoenfeld and Jeremy Kilpatrick pose several major questions: What principles should guide the selection and use of the various tools and processes that are presented in the individual chapters? Where do these tools fit in a “tool space” – the entire collection of tools that mathematics teacher educators might bring to bear in their work? And, more generally, what are the dimensions of proficient teaching? They argue that there is a need for a theory of proficiency in teaching mathematics that could be used to guide the selection and use of tools for mathematics teacher education.

Schoenfeld and Kilpatrick offer, in their chapter, a provisional framework consisting of a set of seven dimensions of proficiency for teaching mathematics: Knowing school mathematics in depth and breadth; knowing students as thinkers; knowing students as learners; crafting and managing learning environments; developing classroom norms and supporting classroom discourse as part of “teaching for understanding”; building relationships that support learning and reflecting on one’s practice. They then elaborate on these dimensions and examine the various chapters in this volume with regard to how the tools they offer can contribute to the framework. This innovative contribution to the domain of mathematics teacher education is viewed, by both authors, as the first steps toward a theory of proficiency in teaching, a theory that provides an orientation to a domain and specifies the skills that people need to develop if they are to become proficient.

CENTRAL ISSUES

The 14 chapters in this volume differ in many aspects: They are written by authors from different cultures, draw on various, theoretical frameworks and describe diverse tools and processes. Still, there are some central issues that most chapters struggle with.

In my own reading of each of the chapters, I found it useful to attempt to formulate responses to the following ten questions:

1. What are the definitions of the tools that are described in this chapter?
2. What are the unique and the general characteristic of these tools?
3. Are these tools valuable for prospective teacher education? For what purposes?
4. Are these tools valuable for practising teacher education? For what purposes?
5. What are the limitations of these tools?
6. What are the suggested ways of utilising these tools in prospective teacher education?
7. What are the suggested ways of utilising these tools in practising teacher education?
8. What is the role of the mathematics teacher educator, when implementing these tools?
9. Are these tools similar to other tools that I know? In what ways? What are the pros and cons of using each of these tools?

10. How relevant and significant are these tools to my own work?

Each of you will probably formulate different questions and different responses while reading the chapters. Please do not hesitate to share these and other issues related to the chapters with me and with the other authors of the chapters.

AN EPILOGUE

The introduction attempts to provide the reader with a preliminary flavour of the book. Many issues and themes are embedded in each chapter. At this point I will let the authors of the chapters tell their own story, so that the reader may explore the variety of tools and processes that interweave to form the fabric of this volume.

SECTION 1

CASES AS TOOLS IN MATHEMATICS TEACHER EDUCATION

OLIVE CHAPMAN

1. NARRATIVES IN MATHEMATICS TEACHER EDUCATION

This chapter deals with the use of narrative, considered from a literary and a humanistic perspective, as a research and pedagogical tool in mathematics teacher education. It presents the methods of obtaining and analyzing stories used in studies of mathematics teacher education. However, the main focus of the chapter is on narrative as a pedagogical tool with prospective teachers in mathematics education. In presenting this use, eight ways in which prospective secondary teachers can be engaged in learning through narrative are given along with example stories.

NARRATIVE AND TEACHER KNOWLEDGE

Interest in narrative, based on the perspective discussed in this chapter, has grown within the field of education beginning in the early 1990s, mostly because of the link advocated between narrative or story and teacher knowledge. For example, Clandinin and Connelly (2000) claimed that teachers' knowledge can only be thought of in narrative terms, as "something lifelike, something storied" (p. 316). Earlier, Elbaz (1990) explained, "Story is something implicit in teachers' knowledge, that is, teachers' knowledge in its own terms is ordered by and as story and can best be understood in this way" (p. 34). Carter (1993) added; "Through story, then, teachers transform knowledge of content into a form that plays itself out in the time and space of classrooms" (p. 7). In addition to this storied nature of teacher knowledge, the link to narrative is also associated with teachers' knowing in action (Schön, 1983). As Doyle and Carter (2003) explained, "Much of the practical knowledge teachers acquire from teaching arises from actions in situations – the essential ingredients of story" (pp. 130–131). Thus,

> The action feature of story would seem to make it especially appropriate to the study of teaching and teacher education. Teaching is intentional action in situations, and the core knowledge teachers have of teaching comes from their practice, i.e., from taking action as teachers in classrooms. (Carter, 1993, p. 7)

Story, then, is a relevant form for expressing teachers' practical understandings because teachers' knowledge is event structured and stories would provide special access to that knowledge. Finally, narrative is also linked to the complex nature of teaching in that it "represents a way of knowing and thinking that is particularly suited to explicating the issues with which we deal" (Carter, 1993, p. 6). In

D. Tirosh and T. Wood (eds.), Tools and Processes in Mathematics Teacher Education, 15–38.

particular, a narrative approach is considered to be able to cope with the many ambiguities and dilemmas that emerge from action. Thus, it is useful to capture teachers' perspectives and the situated complexities of their work and classroom practice, which is often messy, uncertain, and unpredictable (Schön, 1983).

These connections between narrative and teacher knowledge suggest that teachers understand their own teaching through a process of storying their teaching lives and that researchers and teachers themselves can benefit from the process and an analysis of such storying. Narrative can provide a basis for studying teacher knowledge and teaching and a basis for teacher professional development and education. This is reflected in the many studies on teacher thinking or knowledge, teacher learning and teaching in the broader field of education. In mathematics education, based on my review of the literature for this chapter, beginning in the mid-1990s, there was a small but growing body of work involving the use of narrative. In this chapter, I briefly discuss the use of narrative in these studies, focusing on those that deal specifically with mathematics teachers. In particular, I consider the perspectives of narrative used in these studies and how narrative was used. I distinguish between studies that use narrative as a research tool and those as a pedagogical tool. Then, I present my work on the use of narrative as a pedagogical tool with prospective secondary mathematics teachers. In doing so, I describe eight ways in which teachers can be engaged in learning through narrative.

PERSPECTIVES OF NARRATIVE

The concept of narrative has been defined in many ways in the research literature depending on the perspective in which it is framed. In this chapter, the focus is on the literary and humanistic perspectives as a basis of tools in mathematics teacher education. From a literary perspective, a narrative is a story that tells a sequence of events that are significant for the author and his or her audience (Denzin, 1989). It has an internal logic that makes sense to the author. It has a plot, a beginning, middle, and an end, where the plot brings together goals, causes and chance within the temporal unity of a whole action (Ricoeur, 1983). It is a "symbolic presentation of a sequence of events connected by subject matter and related by time" (Scholes, 1981, p. 205). While this perspective of narrative focuses on its structure, it is the humanistic perspective that shapes its use in teacher education.

From a humanistic perspective, narrative is a way of specifying experience, a mode of thought, a way of making sense of human actions or a way of knowing (Bruner, 1986; Crites, 1975; Polkinghorne, 1995, 1988; Sarbin, 1986). It is a "symbolized account of actions of human beings that has a temporal dimension" (Sarbin, 1986, p. 3), "a mode of knowledge emerging from action" (Mitchell, 1981, p. x), a reproduction of the "temporal tensions of experience, a moving present tensed between and every moment embracing a memory of what has gone before and an activity projected, underway" (Crites, 1975, p. 26). According to Bruner (1986), narrative is concerned with the explication of human intentions in the context of action. He also describes it as a way of knowing, a way of constructing reality, a mode of thought centred on the meaning of experience, and the primary form by which human experience is made meaningful. In addition, based on Polkinghorne (1988), narrative is a cognitive scheme; a scheme by means of which human beings give meaning to their experience of temporality and personal actions, a framework for understanding the past events of one's life and for planning future actions, the primary scheme by means of which human existence is rendered meaningful, and a meaning structure that organizes events and human actions into a whole, thereby attributing significance to individual actions and events according to their effect on the whole.

Thus, in the context of this chapter, narrative is viewed as a key form through which individuals come to know themselves, construct their lives, and make sense of their experiences. Since the stories we tell reflect who we are and what we may become, they can provide a basis for meaning recovery and meaning construction or reconstruction of our experiences. They can facilitate interpretation and understanding of our experiences and offer a way of recovering, articulating, and understanding the meanings and intentions embodied in our behaviours. In relation to teacher education, then, such stories provide a humanistic basis for understanding teaching, the teacher and teacher learning.

Based on the preceding literary and humanistic perspectives, narrative has been used in some particular ways in education generally. It has been used as:

– A research method in which the process and product involve stories of teachers and researchers' experiences;
– A tool for collecting data on, for example, teachers' knowledge, beliefs, attitude, life story, and practice;
– An object for analysis in studying teaching;
– A basis for, or tool in, teacher professional development or teacher education;
– A basis for reflective thinking.

In relation to teacher learning, narrative practices are intentional, reflective actions in which teachers write and/or tell stories and work with others (peers, colleagues, other educators, and researchers) to interrogate their past, present and future experiences with teaching as a means of learning through these stories (Connelly & Clandinin, 1988; Johnson & Golombek 2002; Lyons & LaBoskey, 2002). Thus, narrative practices not only provide opportunities for practicing or prospective teachers to share what they know; they can also furnish the means for changing and developing knowledge and help teachers to construct and reconstruct their practical knowledge. Narratives can provide opportunities to help teachers come to know what they do not know and need to know about teaching and learning. They can be used to foster reflective thinking, an important skill for both learning and teaching.

These views of narrative have provided a basis for an increased use in mathematics teacher education. A review of the research literature for this chapter suggests that, beginning in the mid-1990s, narrative was becoming a valuable method for studying and making sense of mathematics teacher knowledge, growth and practice. The different uses of narrative noted above have been adopted in these studies to some degree; however, most of the work involves the use of narrative as a research tool, and a few treat it as a pedagogical tool. The use of narratives as a pedagogical tool, with the exception of my work (e.g., Chapman, 2002), has rarely been explored as a means for enhancing prospective teacher learning. This does not refer to case studies (Chapter 2 of this volume) or video recordings (Chapter 3 of this volume) that could include a different form of narratives. Such narratives usually are not written by, nor do they deal with the perspectives of, the teachers who are using them as a basis of their learning. In this chapter, narratives refer to the self-authored, written or oral stories, which allow teachers to present their teaching or learning experiences from their perspectives and to weave descriptions of their thinking, feelings, attitudes, and other personal attributes relevant to these experiences into the stories. In the remainder of this chapter, I discuss studies that deal with this form of stories in order to highlight the state of the art of narrative as a tool in mathematics teacher education. This discussion is organized in themes based on the two broad uses of narrative identified in these studies: narrative as research tool and narrative as a pedagogical tool. The former considers studies of mathematics teachers' thinking, knowledge, and practice that have implications for mathematics teacher education, while the latter considers studies of mathematics teachers' learning.

NARRATIVE AS RESEARCH TOOL

Studies dealing with the mathematics teacher used narrative as a research tool for investigating several topics (Table 1) in order to produce information on teacher knowledge, thinking, and practice that informs teacher education. Although there are similarities in the use of narrative in these studies, there are differences, not only in relation to the purpose, but also the methods of obtaining and analyzing the stories. A brief description of these methods for each of these studies is given to highlight the uses of narrative as a research tool in mathematics teacher education.

Table 1. Examples of topics studied through narrative methods

Development of prospective teachers' beliefs about, and emotions towards, mathematics	Kaasila, 2007a, 2007b
Why prospective teachers are highly motivated in a social constructivist mathematics course	Harkness, D'Ambrosio, & Morrone 2007
Teachers' interpretations and implementations of a reform-oriented mathematics curriculum	Drake, 2006a, 2006b
Prospective teachers' emerging identities as mathematics teachers and beliefs about their roles as mathematics teacher	Lloyd, 2006, 2005
Challenges to teachers' professional knowledge posed by classroom activities involving mathematical investigations	Ponte, Segurado, & Oliveira, 2003

Teacher change in implementing a project-based curriculum	Ziegler, 2003
Teachers' growth in teaching mathematics	Chapman, 2002; Fijal, 1995
Connecting theory and reflective practice	Smith, 2003
Teachers' perspective of teaching problem solving	Chapman, 1997
Teachers' experiences with implementing reform teaching	Schifter, 1996a, 1996b

Obtaining the Stories

In some of the studies, researchers used an interview approach to obtain the participating teachers' stories and participants were encouraged to tell stories of their teaching or learning experiences. In Kaasila (2007a, 2007b), during the interview, each participant told her or his mathematical autobiography. In Drake (2006a, 2006b), a mathematics story interview protocol was adapted from a more general life story interview protocol developed for use in personality psychology research. In the interview, teachers' narrative descriptions of themselves as learners and teachers of mathematics were collected. The teachers were asked to consider all of their experiences learning, teaching, and using mathematics (both in and out of school) as a story and to identify several key events within that story. These events included the high point, low point, and any turning points in the story. Teachers were also asked to describe significant challenges in their stories and how those challenges were overcome, influential characters in the stories, and possible positive and negative futures, or next chapters, for their stories. In my studies (e.g., Chapman, 2002; 1997), during interviews, participants were prompted to tell stories about their thinking or teaching. Examples of these prompts were:

– Do you consciously try to pass on this view of mathematics to your students? How? Tell a story of a specific situation or event in your teaching that shows this.
– Tell a story of a specific situation or event that involved you as a student in a mathematics classroom when the topic was about ….
– Tell a story of a complete mathematics lesson that involved you teaching (a specific topic) to your students.

While prompts like these were predetermined based on the goal of the study, others emerged during the interviews, in particular, as a basis for teachers to support any theories or generalized claims they offered in telling their stories. The teachers were usually briefed at the beginning of the interviews that stories should describe a specific situation or event as they lived through it. So they should avoid causal explanations, generalizations, or abstract interpretations and, instead, describe the experience as it happened, including feelings/emotions, and thoughts in action, as applicable.

In contrast to collecting the stories during interviews, most of the studies reviewed obtained narratives through writing, i.e., the participants were required to write their stories with various degree of guidance from the researchers. In Harkness, D'Ambrosio, and Morrone (2007), the prospective teachers wrote mathematical autobiographies in a mathematics education course at the beginning of the semester. In Ponte, Segurado, and Oliveira, (2003), the stories were drawn from episodes occurring in classes conducted by the teacher in the project. The teacher wrote stories about situations that occurred in classes where pupils were working on mathematical investigations in order to capture aspects of dilemmas, uncertainties and other elements of her professional knowledge. Ziegler's (2003) participants wrote stories to capture their experiences implementing and teaching a recently mandated applied, project-based high school mathematics curriculum. The stories included feelings and attitudes about the curriculum and descriptions of teaching experiences. Schifter's (1996a) teachers were working with constructivist methods and principles to transform their own mathematics instruction, largely along the lines of the National Council of Teachers of Mathematics (NCTM) Standards. The teachers wrote stories that detailed their successes and failures as they proceeded, with guidance, to reshape the teaching and learning processes in their classrooms. Schifter explained,

> The authors vividly narrate students' words and gestures, bringing readers into their classrooms to "see" and "hear" for themselves. But unlike videotape, this medium presents scenes from the teachers' perspective, complete with their thoughts, doubts, frustrations, and second thoughts. Thus, their audience comes to share the dilemmas they face, the decisions they make, and the satisfactions they experience. (1996a, p. 6)

In Schifter (1996b), the focus of the stories shifted to the teachers' struggle to become a new kind of practitioner as they faced the challenge of new mathematics pedagogy. Finally, Fijal (1995) wrote autobiographical stories of his past and current practice to identify and understand the nature of the changes that occurred over his career as a mathematics teacher. While all of the preceding studies involved telling or writing stories based on actual experiences with mathematics teaching or learning, in Lloyd's (2005, 2006) studies, the teachers wrote fictional stories about mathematics classrooms; in particular, classroom situations and scenarios in the role as the teacher. These stories had as an explicit goal to allow the teachers to create images of the self and others that they may or may not have personally experienced.

Analyzing the Stories

Most of the studies focused on identifying themes within and across stories as a central feature of the data analysis. In order to arrive at the themes, some studies used open coding of the stories as the analysis process, searching for characteristics that related to specific research questions. Ziegler (2003), for example, focused on identifying themes to describe change by first coding key statements in the stories that were indicative of the teachers' practice and philosophies of teaching and learning before and after their implementation of the new applied, project-based curriculum. Fijal (1995), in his autobiographical study, also reflected on his stories to identify patterns of change in his practice and the underlying basis for the change. For Harkness, D'Ambrosio, and Morrone (2007), the focus was on identifying themes in the prospective teachers' autobiography as part of their analysis to examine why these students were highly motivated in a social constructivist mathematics course in which the course instructor emphasized mastery goals. In Drake's (2006a) study, the teachers' stories of their prior and current experiences with mathematics were used to identify and understand patterns in teachers' interpretations and uses of reform-oriented curriculum materials. Drake (2006b) further explained that the process also involved categorizing the teachers' mathematics life stories into six types based on the teachers' descriptions of both their early experiences with mathematics and their current perceptions of themselves as mathematics learners and teachers. The focus of the analysis was on the sense-making practices (noticing, interpreting, implementing) of teachers who provided turning-point stories for the purpose of understanding teachers' specific practices in the context of reform.

For other studies, the analysis process was guided by a predetermined model or specific categories based on theory. Unlike the research previously discussed that focused only on the content of the stories, these studies analyzed both the structure and content of the stories. For example, Kaasila (2007a, 2007b), in analyzing the content of the stories, used emplotment to construct a retrospective explanation of how the participant's experiences at school were reflected in the development of her mathematical identity and in how she taught mathematics during teacher education. In analyzing the form of the story, Kaasila looked at the ways in which the participant told her story, using linguistic features to identify core events in the accounts. Lloyd (2005, 2006) also analyzed the stories structurally and thematically. Rather than attending immediately to the content of the story, the initial analysis of the stories involved using the structural elements to focus on how the prospective teachers represented and interpreted classroom issues and events. In the structural analysis, each story was coded according to a perspective that considered the following narrative elements: abstract (summary of the substance of the narrative), orientation (time, place, situation, participants), complicating action (sequence of events, problems), evaluation (significance and meaning of the action or problem), and resolution (what finally happened). Particular attention was devoted to the structure and meanings of the complicating actions and resolutions in the stories. To help this interpretation, thematic analysis of the content of the stories involved identifying and organizing information about the roles of teachers, students, and mathematics in the plots. Together, these three categories served as a framework for the use of thematic analytic strategies to identify major patterns in the data. Ponte, Segurado, and Oliveira (2003) used the same structural narrative elements as Lloyd (2005) in their first level of analysis of the stories. They tried to identify the main components of a narrative: abstract, orientation, complication, evaluation, resolution, and *coda.* In the second level of analysis, they looked for other issues in the content of the stories that appeared to be significant in the stories.

This section of the chapter highlighted ways narrative was used as a research tool in studies that has implications for mathematics teacher education. This tool allows us to obtain and unpack teaching and learning phenomena in order to understand mathematics teachers and their practice from a humanistic perspective. Studies framed in this perspective focus less on identifying deficiencies in teachers' behaviours and knowledge and more on understanding the nature of, and contexts that shape, and their perceptions of reality. This includes understanding teachers from their own perspective; how particular individual teachers understand their work (e.g., how do teachers make sense of implementing practices of mathematics reform). Thus, a focus of these studies is conceptualizing the experiential knowledge of teachers and providing plausible explanations of teaching behaviours as they are for the teacher. This indicates that using narrative as a research tool can lead to ways of making sense of mathematics teacher education that embody the teachers' perspectives based on their past, present and future experiences. Although narrative is effective as a research tool, it also can be a powerful pedagogical tool in mathematics teacher education. Yet, this use of narrative as a pedagogical tool has been highly overlooked in research in mathematics teacher education.

NARRATIVE AS PEDAGOGICAL TOOL

Narrative can be used as a pedagogical tool in both practising mathematics teacher development and prospective mathematics teacher education. From a theoretical perspective, it can allow both groups of teachers to gain insights into their thinking and actions and deepen their understanding regarding the nature of mathematics and it teaching and learning. It can lead to self-awareness of, for example, what they value, what they would like to do and why they want to do that. It can provide opportunities for them to focus on particular instances of teaching and to examine those instances more deeply than they are able during the busyness of their classroom work. However, while there is a growing interest in using narrative, outside of mathematics education, as a basis of teacher learning (e.g., Blake, Blake, & Tinsley, 2001; Doecke, Brown, & Loughran, 2000; Drake, Spillane, & Hufferd-Ackles, 2001; Goodwin, 2002; Hooley, 2005; Johnson, & Golombek, 2002; Olson, 2000), this is not reflected in mathematics education in terms of published work. Thus, the discussion that follows will be based on my use of narrative as a pedagogical tool.

My work with practising teachers (Chapman, 1999b) provides an example of using narrative inquiry as a basis of their development. The approach includes a focus on telling and reflecting on oral stories involving the teachers' own teaching in order to understand their sense-making of teaching problem solving and to facilitate their extension or reconstruction of this sense-making. Barnett (1998) also reported on a professional development process that used teacher-authored narratives about actual classroom experiences as a stimulus for discussing mathematical, pedagogical and philosophical concepts and issues. The discussion focused on four pivotal areas: development of one's own understanding of mathematics; use of the student perspective as a source of feedback; a recast of the familiar as strange and the simple as complex; and critical examination of alternative views and ideas. On a theoretical basis, Mason's (2002) work on 'noticing' provides several useful guidelines that are applicable to practising teachers' use of narratives as a basis of professional development. In my experience, narrative can be used in similar ways for both practising and prospective teachers. The main difference is obviously the experience of the practising teachers that provides them with a rich repertoire of stories of their own practice. They can thus write stories that include more depth in terms of their thinking, see more possibilities emerging from the stories for discussion and possibilities for the future, and engage in oral story telling more effectively. However, my focus in the remainder of this section will be only on prospective teacher education, where most of my experience lies.

In addition to my work, discussed later, the following studies have indirectly reported on engaging prospective mathematics teachers in narrative. Lloyd (2006) had prospective secondary teachers write fictional accounts about mathematics classrooms. However, her focus was not on discussing or examining this activity from a learning perspective. Smith (2006) asked her prospective mathematics teachers to write stories based on their practicum experiences, but focused more on her self-study of her practice than the stories as a way of knowing for the teachers. Earlier, Smith (2003), in her case study to illustrate connecting theory and reflective practice through personal theories, indicated that she engaged her prospective mathematics teachers in writing reflective narratives for the explicit purpose of thinking about beliefs and actions. She concluded that the use of narrative stories in the form of personal theories can provide an innovative pedagogical tool in mathematics teacher education. In general, while these studies do not provide details on, or directly investigated, the nature and use of narrative in teacher learning, they imply that narrative can play an important role in mathematics teacher education.

In the remainder of this section, I draw on my work with narrative in teacher education to illustrate possible ways in which it has and can be used. In the last decade, I have used stories in a variety of ways as an integral aspect of my mathematics education courses with prospective secondary mathematics teachers (e.g., Chapman, 1999a). I have engaged the prospective teachers with narrative as a reflective process, to foster self-study, and to construct or broaden their pedagogical knowledge. The self-study usually required them to inquire into their thinking about, and to consider, the nature of mathematics and teaching and learning mathematics. In this work, I have used narrative as object and process of inquiry. The following discussion of it is organized under two themes: writing the narratives and narrative reflection/analysis/inquiry.

Writing the Narratives

The narrative process begins by having the prospective teachers engage in intentional storying of past, present and/or future events they have experienced or expect to experience, directly or indirectly. The teachers could tell instead of write their stories. I choose to have them write because it provides focus and attention that simply telling does not. There are several situations that can be used as the bases for these stories. My focus has been on the experiences influencing the development of prospective teachers' beliefs and knowledge about mathematics and its teaching and learning. In particular, I have worked with two categories of experiences: teaching and problem solving. The former produces stories about teaching mathematics, the latter stories about mathematical problem solving.

Stories about Teaching

While this could be open-ended, I have used different situations to provide a focus for the stories about teaching. For example, I had prospective secondary mathematics teachers write stories about:

– the teaching of their own teachers when they were students of mathematics;
– their own teaching during their practicum;
– the teaching of others (i.e., peers, cooperating teachers, other teachers) observed during their practicum; and
– their teaching as imagined. This is a narrative equivalent of a lesson plan that allows them to imagine the lesson as they perceive or desire it to actually unfold.

These situations can be presented to the prospective teachers as open, i.e., they choose what they want to write about, or specific, e.g., a situation that they consider to be "good teaching" of mathematics, a situation they consider to be "bad teaching" of mathematics, or a memorable mathematics lesson they experienced. I have used the specific case because, from experience, the open case often produced stories that ignored the mathematics, as further explained later in this section. The stories can be written on any number of these situations as single activities or as a combination of two or more in ways that build on each other depending on the depth with which the use of narrative is being integrated into a course. For example, at the beginning of a mathematics education course, I had the prospective teachers write three stories they had experienced as a student, a teacher or an observer – one that they considered to be "good teaching" of high school mathematics, one that was "bad teaching" and one that was "memorable." The intent of this activity was to capture their preconceptions of mathematics pedagogy on entering the course.

The assumption underlying having the prospective teachers write such teaching stories is that if they are viewing or thinking about mathematics teaching through their preconceptions, then the way they recall or imagine a mathematics lesson will be influenced by this filter. Thus, they write themselves into the situations they describe, either literally or as the observer/author who defines what matters most. This means that in writing the stories, the focus is not on the accuracy of how a past situation or a mathematics lesson unfolded, but, for example, what mathematics lesson they choose to describe, what they choose to recall about it and how they tell it. As a result, they can fill gaps in their memories in a way that makes sense to them. As Polkinghorne (1988) explained:

> Narrative configuration is not simply a personal projection that has no relation to worldly events … it is required to attend to the accepted reality of those events. Never the less, narrative meaning consists of more than the events alone; it consists also of the significance these events have for the narrator in relation to a particular theme. (p. 160)

However, as mentioned before, it may be necessary to provide the prospective teachers with specific guidelines and boundaries for the stories depending on their intended purpose.

The instruction given to the prospective teachers can influence the nature of the stories they write and the usefulness of the stories in addressing mathematics pedagogy as opposed to generic pedagogy. For example, I found that simply telling them to "write a story about …" did not produce the type of stories I expected, i.e., with emphasis on mathematics pedagogy. In my early years of beginning to use narrative, asking the prospective teachers to write a story about their teaching of mathematics during their practicum, with an explanation of what I meant by story, produced stories with little or no attention to mathematics pedagogy. Instead, they wrote about classroom management issues or positive rapport with particular students; two of the traditional concerns to them – controlling the class and being liked by their students. It took a few iterations of modifications by adding details before arriving at a written guideline that produced not only powerful stories of mathematics teaching, but also a process that engaged the prospective teachers in deep self-reflection. The following is an example of this guideline used at the beginning of the course before they are exposed to any theory on mathematics pedagogy.

Write a story of a mathematics lesson you experienced as a student or an observer that you consider to be "good teaching" of high school mathematics. The story should describe at least one complete mathematics lesson from beginning to end and provide as much details as possible on the following: what the teacher did and said; what students did and said; how the mathematics content was dealt with or presented. The lesson should involve engaging the students in a mathematics concept for the first time (i.e., not a review or practice lesson). The story should be written in present tense and include direct speech of teacher and student. Do not analyze anyone or anything in the story. Just describe the situation as you think it actually happened. Do not leave gaps in the story, i.e., if you cannot recall a specific detail in exactly the way it happened, describe it based on what makes sense to you. But it must be plausible.

In general, the first sentence of this guideline can be modified to create variations in the stories and what the teachers are prompted to reflect on. In the case of imagined stories, I replace it with: *select a grade and a mathematics topic/concept you expect to teach and write a story of how you will like to teach it.*

Stories about Problem Solving

Like the preceding category of teaching stories, for this second category of stories I used in my teaching, I also provide different situations as a focus for the stories. In this case, the situations involve problem-solving experiences. For example, I had the prospective secondary mathematics teachers write stories about:

- their problem-solving experience based on a prescribed problem,
- the problem-solving experience of others, usually secondary students or peers, and
- their experience presenting a problem to others, usually peers.

The goal for this story-writing task is for the prospective teachers to inquire about problem solving and their role as teacher.

Similar to the case of the teaching stories, the guidelines I developed to elicit appropriate stories from the prospective teachers evolved in reaction to what they would produce when asked to solve the problem and write a story describing the experience. They would write about how they thought they solved problems in a theoretical way and ignore the experience of doing the assigned problem. They would focus on describing only the steps to the solution and ignore the affective features of the experience. These behaviours suggested the type of modifications that were necessary to develop written guidelines that produced genuine narratives of the experiences involved. The following example is the guideline for the first of the three situations noted above for generating stories about problem solving:

Write a narrative of your experience solving the problem that is attached. The narrative should include your complete solution, feelings/emotions, thought processes, (in general) all actions associated with doing the problem from the moment you read it, so do not read it until you are ready to work on it!!! A page of "steps" to getting a solution is unacceptable and you will be required to re-do the narrative with a different problem. Correct method or answer is not important, but there should be a serious attempt at getting one for the narrative to be meaningful. Your focus should be on describing your personal experience in doing THIS problem. All of your mathematical work in arriving at a solution should be included or attached.

Non-routine mathematical problems are used for this activity. In particular, problems are selected to allow the prospective teachers to experience being stuck and getting out of being stuck.

The stories usually vary from two to five single-spaced pages depending on the individual prospective teacher's experiences. The process of creating the narratives is rich with opportunities for the prospective teachers to engage in reflection. In the case of the teaching stories, choosing what details to include, remembering the conversations that occurred, thinking back on the feelings that were part of the event, remembering who did and said what, deciding on what is good or bad teaching, and so forth, are all parts of the writing process that spurs the prospective teachers to reflect on their thinking and how teaching is or can be lived in the mathematics classroom. In fact, they usually find writing the initial stories challenging because they never had to articulate their thinking about, and experience with, mathematics pedagogy in this holistic and fine-grain way. In the case of the problem-solving stories, the prospective teachers have to attend to, and notice details of, the experience involved beyond their preconceptions of problem solving. In general, then, writing the story itself becomes a reflective and a learning activity because the prospective teachers are forced to attend to details of practice and problem solving they tend to dismiss. Once the stories are written, they form the basis of further reflection, analysis or inquiry, discussed after the following examples of the stories.

Examples of Stories

These examples of teaching stories were written by prospective secondary teachers as part of their course-work in mathematics education. They provided a basis for exploring both mathematics content and pedagogy. Both stories were based on actual lessons the authors had previously observed.

Example 1

This is one of the shorter stories. It is in response to the prompt to write a story that dealt with the treatment of a mathematics concept that raised issues for the author during a lesson he or she observed. The author also posed questions in regard to these issues for group discussions with peers. The story was written three-quarters into the semester. It is based on a Grade 12 lesson.

A teacher I observed was giving a lesson on Ellipse (conic). After defining ellipse as a conic generated when a plain intersects a cone (or a right cylinder) at a certain angle, he proceeded to develop the concept of ellipse by applying a series of transformations on the unit circle $x^2 + y^2 = 1$.

He explained: "If we do a horizontal stretch, say, by a factor 2, and a vertical stretch by a factor 3, the equation $x^2 + y^2 = 1$ *becomes* $\frac{x^2}{4} + \frac{y^2}{9} = 1$. *If you recall in our first unit, to indicate a horizontal stretch, you multiply x by the reciprocal of the horizontal stretch factor. That is why we have the 4 under* x^2. *Since y is implicit – take note, we are not dealing with a function here but a relation, so instead of multiplying y by 3, we also multiply y by the reciprocal of the vertical stretch factor."*

I saw some students' heads nod in agreement. Also, there were some students who were sitting at their desks with a stoic expression.

The teacher continued, "Now let's translate the new graph, say, 5 units to the left and 6 units up. As what happens with a circle, the center of the new graph becomes (-5, 6). Therefore, the equation becomes,

$$\frac{(x+5)^2}{4} + \frac{(y-6)^2}{9} = 1$$

In general, if we stretch a unit circle with center at (0, 0) horizontally by a factor a and vertically by a factor b, and at the same time horizontally translate it h units and vertically k units, the equation $x^2 + y^2 = 1$ *becomes*

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The graph of this equation is called an ellipse and this equation is called the standard form of the equation of an ellipse.

Note that as in the circle, (h, k) is the center of the ellipse. The translation moved the center (0, 0) to the point (h, k). Consider the following example:

A unit circle whose center is at the origin has undergone the following transformations:

Vertical stretch by a factor 10

Horizontal stretch by a factor 8

Translated to the right 6 units and down 3 units.

The new graph is called an ellipse and its equation in standard form is:

$$\frac{(x-6)^2}{64} + \frac{(y+3)^2}{100} = 1."$$

The teacher gave two more examples in determining the standard form and then asked the students to try a few exercises such as the following:

Using the standard form, write the equation of an ellipse obtained from $x^2 + y^2 = 1$ *by:*

Vertical stretch by a factor ½

Horizontal stretch by a factor 7

Translation of 1 unit right, 2 units down.

There were several students who wrote,

$$\frac{(x-1)^2}{49} + \left(\frac{1}{2}\right)(y-2)^2 = 1$$

A response like this was not uncommon for several days to come, up to the time the lesson on conics was concluded.

The following questions were posed by the author for group discussion:

– *How do you make sense of the students' answer to the indicated exercise above?*
– *Would you agree with how the standard form of equation for ellipse was developed/presented? If not, how would you do it?*
– *The first unit was on the transformation of the function y = f(x). The students were taught that a horizontal stretch of 2 units, vertical stretch of 3 units, translation 1 unit to the right and 3 units up on the graph of y = f(x) would yield a new function:*

  $$y = 3f\left(\frac{x-1}{2}\right) + 3.$$

  Knowing this, would it change the way you were to develop the standard form of the ellipse equation? If so, how? Or, would you change the way the concept of transformation was done in the previous unit? How?

Example 2

This example is based on the guideline, discussed earlier in the chapter, for writing a story of "good teaching". It was written at the beginning of the semester before the author had any exposure to theory on mathematics pedagogy, so it is based solely on her preconceptions of "good teaching". It is based on a Grade 10 lesson at the beginning of a trigonometry unit.

The sound of music began as the bell ended and the students filed into the classroom. The teacher greeted most of the students as they walked in. The teacher also makes note of a few students' new haircut or new shirt and compliments them. The teacher waited a little bit for the students who trailed in just after the sound of the music stopped. The students quickly settled in their seats. The teacher walks to the front of the class.

Teacher: Okay guys, today we're starting a new unit, Trigonometry. Does anyone remember what Trig is, from last year?

The teacher looks around the room and paused to give the students time to think. A student's hand went up.

Student: Is it that thing with SOH CAH TOA?

Teacher: Yes! SOH CAH TOA is definitely part of trig. Great job. Does anyone else remember doing trig before?

Student: Is this the unit with the triangles and angles and $a^2 + b^2 = c^2$*?*

Teacher: Yes this is definitely that unit. So most of you guys have seen trig before, so this should be a nice little review to refresh your minds and some of you may have not seen this concept before which is okay. In this unit we're going to learn about triangles and how we can get the measurement of the sides and angles of the triangles. We're also going to look at how we can use those in the real world. Near the end of this unit we have a fun project to do where we are going to use what we learned to measure the distance from the school's floor to the ceiling in the main hallway.

The teacher turned to the white board, marker in hand, and started to write notes for this unit. As she writes, she reads the words out loud. Some students already have their notebooks and pencil ready whereas some other students are just taking them out of their bags.

Teacher (talking and writing): Unit 7 – Trigonometry

Teacher: So the first thing we're going to look at in this unit is right angle triangles. Can anyone tell me what makes a right angle triangle?

A student raises his hand. The teacher calls on him.

Student: One of the angles has to be 90 degrees.

Teacher: Good job, you're completely right. So class, when we're looking at triangles, if one of their angles in 90 degrees then we know that it is a right angle triangle.

The teacher turns to the board.

Teacher (talking and writing): Right angle triangles are triangles that have a 90-degree angle.

The teacher draws a picture of a right angle triangle on the board, indicating a square in the corner which represents the angle that is 90 degrees.

Teacher: Now we're going to look at a theory called Pythagorean Theorem. This was mentioned earlier by Jim.

Teacher (talking and writing): Pythagorean Theorem $a^2 + b^2 = c^2$

Teacher: This theorem can only be used for right angle triangles and we use it when we know two sides of a triangle and we are looking for the remaining unknown side.
She then draws a picture of a right angle triangle and asks the class: Does anyone think it matters what letter I label for each side of this triangle?
A student raises his hand. The teacher calls on him.
Student: Does one of the letters must always be the hypotenuse or something like that?
Teacher: Yes, you are absolutely right. Class, when we're using Pythagorean Theorem it is very, very, very important that we label c as our hypotenuse. This is probably the most important part of using this theorem.
The teacher turns to the white board and starts labelling her triangle. She labels c at the hypotenuse. She then asks the class: Do you think it matters which side I label my a and b?
Student shouts out: No.
Teacher: That's right. It really doesn't matter which side I choose for my a and which side I choose for my b, so long as I always, always label my c as the hypotenuse. Do you guys know how you would know which side is always the hypotenuse?
Student: It's the biggest side out of all the sides.
Teacher: Yes, you're absolutely right Tracy; the hypotenuse is always bigger than the other two sides. You can also know which one is the hypotenuse because this side is always across from the 90-degree angle.
The teacher points at the drawing and points at how the hypotenuse is always the side across from the 90-degree angle. The teacher labels a and b on the triangle. The teacher draws an arrow that points at the c on the hypotenuse and writes: c must be the hypotenuse.
Teacher: Okay, let's do two examples using $a^2 + b^2 = c^2$ *and then I'll pass around a worksheet for you guys to practice this.*
Teacher (talking and writing): Example 1: Find the missing side.
The teacher draws a picture of a right triangle, labels one side 4 and another side 6, leaving the hypotenuse empty.
Teacher: Okay, so who can tell me what I know in this example?
Student: We know that one side is 4 and the other is 6.
Teacher: Can you tell me which side we are looking for?
Student: The hypotenuse.
Teacher: Great job, Kim. Now who can help me fill in the formula $a^2 + b^2 = c^2$*?*
Student (as teacher writes): Put $4^2 + 6^2 = c^2$
Teacher: Great job. Now can you guys tell me what $4^2 + 6^2$ *is?*
Student: $16 + 36 = 52$
Teacher writes on the board: $16 + 36 = c^2 \quad 52 = c^2$
Teacher: Are we done?
Some students shout out yes and some shout out no.

Teacher: No, we're not quite done yet because right now we have c^2 but we want only c, so we're going to have to square root both sides to get c. Can you guys square root 52 in your calculators and tell me what you get.
Students: 7.2
The teacher writes on the board: $\sqrt{52} = c \quad 7.2 = c$
Teacher: Okay guys great job! Let's do the last example.
Teacher (talking and writing): Example 2: Find the missing side.
The teacher draws a picture of a right angle triangle, labels one side 5 and the hypotenuse 8, leaving one side empty.
Teacher: Okay, so can someone help me fill in the formula?
Student (as teacher writes): Put $5^2 + b^2 = 8^2$
Teacher: Great job. How did you know to put 8 as c?
Student: Because c must be the side across for the right angle.
Teacher: Great job. So class does everyone see how in this example, we must put 8 as c because 8 is the hypotenuse which is the side across from the 90 degree angle? Okay can someone tell me what 5^2 *is?*
Student: 25
Teacher writes on the board: $25 + b^2 =$
Teacher: And who can tell me what 8^2 *is?*
Student: 64
Teacher writes on the board: $25 + b^2 = 64$
Teacher: Now this question is a little trickier than the first example because we don't have b by itself. So we're going to have to get b by itself. Can anyone tell me what you think I should do to get b by itself?
Student: Subtract 25 on both sides.
Teacher: Yup, that's right.
The teacher writes on the board:

$$\begin{array}{rcl} 25 + b^2 &=& 64 \\ -25 & & -25 \\ b^2 &=& 39 \end{array}$$

Teacher: Are we done?
Some students shout out yes and some shout out no.
Teacher: No we're not done yet because right now we have b^2 *but we want only b, so we're going to have to square root both sides to get b. Can you guys square root 39 in you calculators and tell me what you get?*
Students: 6.2
Teacher writes: $b = 6.2$
Teacher: Did everyone get the same answer?
The students agreed.
Teacher: OK! Well that's all the notes we're going to take for today, I'm going to pass around a worksheet for you guys to do for the remainder of the class working on questions using Pythagorean Theorem (their worksheets for the week is always due on Monday of next week). Tomorrow we'll start to look at how to find angles in a triangle.
The teacher passes out the worksheet to the students one by one and goes back to her desk and does attendance in her attendance book, then on the electronic

system. The students are now getting into gear and into their worksheet. Some work alone, and some work in groups. When the students have questions, some come up to the teacher's desk and some raise their hands and the teacher comes over to them. They continue to work on their worksheet until the period is over. The teacher walks around the room helping students. Near the end of the period, the students start to pack up and wait around the door until the music in the hallways comes on. Once the music comes on, the students head off to their next class.

Reasons the author gave for this being "good teaching" included: Classroom management was very good. The students listened well and were respectful. The teacher taught the concept in a good manner to the students, the concept was taught correctly, and examples were used to show the concept. Later in the semester, the author used the NCTM (1991) standards on discourse, worthwhile tasks and environment to analyze the story. Her response reflected a self-realization of the limitations in her preconceptions. For example, what she initially considered as good questions she now did not. She explained:

> The teacher in my story did not pose meaningful questions to these students, questions that allowed them to think beyond what is told to them. … Most questions in the story focused on those that had one right answer.

At the end of the semester, she re-wrote the story to indicate how she would teach the topic. This story now contained features of an inquiry-based, student-oriented approach to teaching. The narrative provided a basis for her to understand the shifts in her thinking and knowledge, which she indicated was reinforcing in terms of the teacher she will like to become. The next section discusses this and other ways in which prospective teachers can use their stories in their learning.

Narrative Reflection/Analysis/Inquiry

Narrative reflection, analysis or inquiry allows the prospective teachers to explore their stories to gain deeper understanding of mathematics pedagogy and problem solving, in my case. The written stories become a basis for inquiring into the meaning of specific themes embodied in them. I summarize eight ways in which I have engaged prospective secondary teachers in this process of narrative reflection/analysis/inquiry. All of them are not mutually exclusive. The intent is to indicate a range of what is possible. Thus, all of the ways do not have to be used in the same course, for example. One can select one or a few or create variations that best fit one's context. For example, I have combined ways (i), (ii), and (iii) in a course on mathematics pedagogy.

(i) Initial self-reflection

In this activity, after writing their teaching stories, the prospective teachers wrote journals on why they chose the stories they wrote, e.g., what made their stories "good teaching". They then read a sample of each other's stories and wrote journals on which of the stories they thought represented good teaching and which stories they liked and did not like in terms of the teaching, giving reasons for their choices. They then shared, compared and discussed their thinking with each other. This initial self-reflection allowed them to confront their preconceptions and beliefs, e.g., resulting from conflicts among them in terms of their views of the stories as good or not good teaching.

(ii) Restorying

For this task, towards the end of the course, after being exposed to theories and other experiences on mathematics pedagogy, the prospective teachers revisited their teaching stories written at the beginning of the course to restory them. They were asked to discuss if their teaching stories were inquiry-based lessons. Usually most of them indicated that the lessons were not and that their understanding of "good teaching" was now different from the initial story. The others, whose stories did include some aspects of inquiry, indicated that their understanding of good teaching was validated and expanded. After this reflection, they rewrote their stories of "good teaching" in the way they would want to see them unfold as inquiry-oriented lessons. Even if they thought their stories consisted of inquiry-based teaching, they had to provide an alternative inquiry-oriented way of conducting them. In rewriting these stories of teaching, they had to make conscious choices about how to conceptualize themselves and their roles. They had to use their imagination, picture the classroom more finely, and consider possibilities for the mathematics and engaging students in learning it. So their rewriting of the teaching story captured the shifts in their knowledge in how they initially viewed teaching and the mathematics.

(iii) Unpacking teaching stories

This activity treated the stories as cases to unpack during the semester as a basis to interpret theory discussed in the course and as a basis for discussing mathematics content in context. The prospective teachers analyzed their stories individually and in groups. The analysis included making sense of: what mathematics was being taught; worthwhile mathematics tasks; how students were engaged in the content; discourse – teacher-student interactions about the mathematics; teacher's role; students' role; and any issues unique to the story or raised by the class (e.g., assessment). This analysis was guided by readings, such as the NCTM (1991) Standards. For example, the prospective teachers had to decide whether or not the mathematical tasks and discourse in their stories were consistent with the characteristics proposed by the NCTM Standards. This analysis of practice broadened their awareness and understanding of details of mathematics teaching they had taken for granted or not noticed. For example, initially, they saw students as being actively involved if they were responding to the teachers' questions, but did not notice that the questions were mostly factual or memory recall. The analysis of the teaching stories also challenged the teachers' mathematics knowledge for teaching and allowed them to think more deeply about the mathematics. In using these stories as cases, they offered the prospective teachers something unique in comparison to more traditional cases. These stories offered the prospective teachers situations of practice that were real to them as a basis of learning about teaching mathematics. These were their stories. They knew the stories' broader and specific contexts, e.g., the teachers, students, culture and schools. This allowed them to deal with the stories as lived experiences as opposed to what may seem as theoretical cases to them if the stories were provided by others.

(iv) Unpacking problem-solving stories

The activities to unpack the problem-solving stories depended on the experience that framed the stories. For example, in the case of the stories that involved the prospective teachers' experience solving an assigned problem, the prospective teachers examined their stories for what they told about problem solving, when and how emotions occurred in the problem-solving process, where the problem solver was stuck and how he or she tried to get out of being stuck, and implications for teaching. This process provided the prospective teachers with a more realistic view of problem solving and more meaningful ways to teach it than they initially held.

(v) Narrative inquiry through peers

This form of narrative inquiry involved using the written story as a stimulus for sharing oral stories of related personal experiences. It required the prospective teachers to work in small groups, sharing and resonating with each other's stories. For example, one person would select and share an excerpt from her or his story, the rest of the group then resonated with it by listening for what they perceived to be similar or different in their own experiences (within or outside of their own written stories), asking questions about it and sharing points of similarity and difference in experiences. This activity allowed the prospective teachers to gain deeper understanding of themselves and to assist each other to clarify her or his understandings.

(vi) Comparing imagined versus actual actions

This activity allowed the prospective teachers to create a bridge between their course work and field experience by comparing their imagined story of how they intended to teach with their actual teaching. For this activity, in the methods course, instead of a traditional lesson plan, each of the prospective teachers created a story, following the guidelines discussed earlier, of how he or she planned to teach a specific mathematics concept during the practicum. This often occurred at least two weeks before they actually taught the lesson. After teaching the lesson they returned to their stories and compared it to what actually happened, focusing on what was similar and different, reasons for differences, and possibilities for the future.

(vii) Comparing personal versus theoretical knowledge

This activity allowed the prospective teachers to use their stories as a basis to compare their knowledge based on their past experiences with theoretical knowledge they were being exposed to in the mathematics education course. For example, they identified the perspectives of mathematics, learning, and teaching portrayed in their stories, then compared and considered the implications of these perspectives in relation to alternative perspectives provided by theory.

(viii) Identifying narrative themes

This final example of activities in which prospective teachers can learn from their written stories involves having them identify and consider a theme from a set of at least three teaching stories based on different teaching situations. This activity provided a way for the prospective teachers to make explicit the meanings underlying their thinking about mathematics pedagogy, and a way of seeing contradictions in terms of conflicting issues encountered in determining the theme.

This section of the chapter highlighted ways in which narrative can be used as a pedagogical tool. These different ways of engaging teachers in narrative as a mode and object of knowing illustrate a range of activities that focus on mathematics teacher education. Formal and informal studies I have conducted on these narrative approaches suggest that they have the potential to enhance prospective teachers' learning. The approaches are learner-focused in that predetermined knowledge does not dominate the learning process; instead, the personal narrative of each prospective teacher forms the direction of the learning experience. The narratives of others ensure that knowledge external to the learner is available. Narrative interactions with peers help to build connections with others in ways that allow and encourage the joint construction of knowledge and to hear and acknowledge multiple perspectives on an experience or idea. However, like any other tool, the usefulness of narrative is dependent on particular factors, including the people and context involved. For example, the prospective teachers need to feel psychologically safe in order to write and share their stories based on actual experiences. Some teachers who dislike writing as mathematics majors are likely to not provide the details necessary for the stories to be meaningful as a basis to reflect on and analyze important pedagogical issues. My experience is that this can be avoided if the guidelines for writing the stories and the purpose for the activities are clearly discussed with them.

CONCLUSION

Narrative as research method seems to have received much more attention in studies related to mathematics teacher education than narrative as pedagogical tool. But stories could provide opportunities to allow prospective teachers to capture lived experiences of mathematics education, to explore their thinking about teaching mathematics, to construct or refine personal theories of learning and teaching and to get to meanings and alternative perspectives that could influence their behaviours in a positive way in teaching mathematics. Narrative provides a reflective way of knowing, which is widely accepted as a central goal in teacher education. It seems that future work in mathematics teacher education should include considerations of narrative as a pedagogical tool. For example, future research can consider: What is the role of narrative in the construction of mathematics teachers' knowledge? What is the meaning and usefulness of narrative to teachers learning to teach, as well as to experienced teachers? However, it is also important to continue to explore narrative as a research method in mathematics teacher education. While this use of it is growing, it is nevertheless very limited. Thus, there is still a lot that could be done with narrative as a pedagogical tool and even more to understand about it as a process and product of research.

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2. CASES AS TOOLS IN MATHEMATICS TEACHER EDUCATION

This chapter explores the use of cases in mathematics teacher education. In particular, notions of what a case is and what teachers learn from their experience with cases is discussed. Drawing on our work we present examples of two different types of cases – exemplars and problem situations – that are examined and discussed in detail. On the one hand, exemplars are used to exemplify a practice or operationalize a theory. They provide vivid images of teachers in real classrooms that ground abstract ideas related to content and pedagogy. On the other hand, problem situations can be used to examine the complexities of teaching and the problematic aspects of performance. They often provide dilemmas to be analyzed and resolved. We conclude the chapter by discussing additional work that is needed regarding the use of cases in teacher education, including the design of additional types of cases and more research that addresses what teachers learn, how they learn it, and the impact of their learning on their professional practice.

INTRODUCTION

Historically, cases were first used at the Harvard Law School more than one hundred years ago (1871), and were subsequently used at the Harvard Medical School (1910) and the Harvard Business School (1920s). Today, cases are used in many other fields as well, such as geography (e.g., Grant, 1997), physical therapy education (e.g., McGinty, 2000) and teacher education (e.g., Merseth, 1991). Case use in teacher education began in earnest in 1986 when Lee Shulman, proposed "case knowledge" as a component of "teacher knowledge" (Merseth, 1996).[1] Although the fields are quite diverse, there appears to be a general belief that cases can address effectively a common tension in the design of experiences for professional education. A professional education curriculum seeks both to provide codified, theoretically based knowledge and to teach reasoning skills and strategies for analyzing and acting professionally in novel settings. Such a curriculum is grounded in the obligation of professional education to prepare practitioners for a practice that is simultaneously routine and uncertain (Sykes & Bird, 1992).

With the growth in the use of cases in a wide variety of fields, many different kinds and types of 'cases' have emerged, calling into question, what is a case? Herreid's (1997, p. 92) definition of cases captures the core element of what makes

[1] Although cases were being used in teacher education as early as the 1920s, these efforts were not as extensive or well organized (Lundeberg, Levin, and Harrington, 1999).

D. Tirosh and T. Wood (eds.), Tools and Processes in Mathematics Teacher Education, 39–64.

something a case: “Cases are stories with a message. They are not simply narratives for entertainment. They are stories to educate.” Hence a case in the field of teacher education is defined as “any description of an episode or incident that can be connected to the knowledge base for teaching that can be interpreted …” (Carter, 1999, p. 174). Merseth (2003, p. xvii) argues “good cases bring a ‘chunk of reality’ into the teacher education classroom to be examined, explored, and utilized as a window on practice …”

In this chapter we discuss cases and their use in mathematics teacher education. The chapter is divided into three sections. In the first section, we provide a rationale for the use of cases in mathematics teacher education and report on what teachers can learn from their experiences with cases. In the second section we provide in-depth examples of two different types of cases – exemplars (i.e., lengthy narratives that portray an instructional episode in its entirety, highlighting the actions and interactions of a teacher and her students), and problem situations (i.e., shorter scenarios that focus on specific problems or dilemmas encountered by teachers as they listen to and interact with students). Section three concludes the chapter in which we discuss key aspects of the use of cases and raise questions about the use of cases.

THE USE OF CASES IN MATHEMATICS TEACHER EDUCTION

The first book of cases for mathematics teacher education was published in 1994 (see Barnett, Goldenstein, & Jackson, 1994) and launched a new era in the education of teachers of mathematics. Since the publication of this volume over a decade ago, many additional mathematics casebooks have been published in English (e.g., Merseth, 2003; Schifter, Bastable, & Russell, 1999, 2002, 2007; Seago, Mumme, & Branca, 2004; Stein, Smith, Henningsen, & Silver, 2000) and in other languages (e.g., Markovits, 2003). These casebooks vary greatly in terms of content focus (e.g., specific mathematical ideas, students’ thinking about particular pieces of mathematics, the pedagogy used to support student learning of mathematics), grade level (e.g., elementary, middle, high school), type (e.g., narrative or video), authorship (e.g., written by teachers describing their own practice or written by a third party describing some aspects of classroom instruction), and authenticity (e.g., a portrayal of events as they actually occurred, episodes based on real events but embellished, or hypothetical situations based on research on teaching and learning). Despite these differences, cases in mathematics education share a common feature of providing realistic contexts for helping teachers “develop skills of analysis and problem-solving, gain broad repertoires of pedagogical technique, capitalize on the power of reflection, and experience a positive learning community” (Merseth, 1999, pp. xi–xii).

Consider, for example, the case shown in Figure 1. This case, written by a middle-school teacher, describes the actual events that unfolded in her classroom when seventh-grade students were introduced to the concept of ratio. The case provides an opportunity for teachers to explore important mathematical ideas related to rational number understandings, to analyze and critically reflect on the teacher's actions and interactions in the classroom, and to consider different courses of actions open to the teacher. By learning how to analyze "messy and complex situations", such as the one presented in the ratio case, teachers can learn to make well-informed decisions in their own classrooms (Barnett & Ramirez, 1996, p. 11).

The Ratio of Boys to Girls

We often hold discussions in my seventh grade class, and sometimes I find myself totally unprepared for the questions my students ask. It's not that I feel I should be the one with the answers, but I do want to guide the discussion productively.

In a recent lesson introducing ratio concepts, I had written the fraction $\frac{1}{2}$ on the board and reminded my students that it meant "1 divided by 2," and that it also meant "1 out of 2."

"It is a division problem and also a ratio because it shows a comparison of two numbers – 1 and 2," I explained. "Let's compare the number of boys and girls in our class." The class determined that there were 17 girls and 15 boys. "What is the ratio of girls to boys?"

Several students called out, "Seventeen to 15." I wrote the fraction $\frac{17}{15}$ on the board.

Carmen blurted out, "That can't be right. You said a fraction means the top divided by the bottom. That'll be more than one whole class."

Laura interjected, "And you said you could say "out of" – like 17 out of 15. That doesn't sound right – 17 girls out of 15 boys."

I realized that I wasn't very clear myself about how fractions and ratios were related or what part context plays in describing ratios. These were good questions and I wasn't sure how to handle them.

Figure 1. An example of a teaching episode.[2]

Hence cases are one way of providing prospective and practicing mathematics teachers with the opportunity to develop knowledge needed for teaching (e.g., knowledge of content, pedagogy and students as learners) as well as the capacity for knowing when and how to apply such knowledge, a capacity that depends on the ability to connect the specifics of real-time, deeply contextualized teaching

[2] Taken from *Fractions, Decimals and Percents* edited by Carne Barnett, Donna Goldenstein, and Babette Jackson.

moments with a broader set of ideas about mathematics, about teaching, and about learning. To develop this capacity, teachers must learn to recognize events in their own classrooms as instances of larger patterns and principles. Then they can formulate ways of acting and interacting that are thoughtful, principled, and effective (Shulman, 1996). As Shulman (1992, p. 28) has noted:

> I envision case method as a strategy for overcoming many of the most serious deficiencies in the education of teachers. Because they are contextual, local, and situated – as are all narratives – cases integrate what otherwise remains separated. Content and process, thought and feeling, teaching and learning are not addressed theoretically as distinct constructs. They occur simultaneously as they do in real life, posing problems, issues, and challenges for new teachers that their knowledge and experiences can be used to discern.

Although it is unlikely that cases alone are sufficient as a source of professional learning (Patel & Kaufman, 2001), they can be a critical component of a curriculum for teacher education, providing a focus for sustained teacher inquiry and investigation (Ball & Cohen, 1999), an opportunity to make connections to experiences (vicarious or lived), and to theoretical classifications and general principles (Shulman, 1996). By deprivatizing teaching, cases can help teachers deal with the uncertainties presented and begin to think like teachers. According to Richardson (1996, p. ix), one challenge of teacher education is to help teachers “begin to develop practical knowledge that will allow them to survive the reality of the classroom.” Cases appear to be one way of facilitating teachers’ development of this practical knowledge.

While there is considerable enthusiasm for using cases in teacher education, and many claims regarding the efficacy of this approach (e.g., Merseth, 1991; Sykes & Bird, 1992), establishing an empirical basis for these claims has been a slow process. In 1999 Merseth noted “the conversations about case-based instruction over the last two decades has been full of heat, but with very little light” (p. xiv). Although much more work is needed, research on the use of cases does provide evidence that they can be used to enhance teachers’ pedagogical thinking and reasoning skills (e.g., Barnett, 1991); help teachers reason through dilemmas of practice (e.g., Harrington, 1995, Markovits & Even, 1999b); support inquiry into classroom practices (e.g., Broudy, 1990); help teachers learn key pedagogical practices that support student learning (e.g., Stein, Engle, Hughes, & Smith, 2003; Hillen & Hughes, in press); and facilitate the development of content knowledge (e.g., Merseth & Lacey, 1993).

One coherent attempt to define an empirical basis for the use of cases in teacher education is the book entitiled, *Who Learns What From Cases and How?: The Research Base for Teaching and Learning with Cases* edited by Mary Lundeberg, Barbara Levin, and Helen Harrrington (1999). The chapters in this book report the findings of a series of studies, mostly descriptive or naturalistic, conducted by the authors in an effort to determine what students enrolled in their teacher education courses learned. The studies, drawing on a variety of data sources (e.g., interviews with teachers, videotapes of sessions, written reflections) and methodologies (e.g., discourse analysis), highlight ways in which cases develop teachers knowledge base and decision making skills but leave many unanswered questions regarding the extent to which cases influence teacher or student performance in the classroom. In general, the studies reported in this book lack objectivity (i.e., the researcher is generally the teacher educator implementing the case instruction that is being studied) and rarely use traditional research designs.

THE ‘WHAT AND HOW’ OF CASES: A FOCUS ON TWO SPECIFIC TYPES

Cases can be divided into two broad categories, *exemplars* and *problem situations* (Carter, 1999). On the one hand, exemplars can be used to exemplify a practice or operationalize a theory. They provide vivid images of teachers in real classrooms that ground abstract ideas related to content and pedagogy. Problem situations, on the other hand, can be used to examine the complexities of teaching and the problematic aspects of performance. They often provide dilemmas (either mathematical or pedagogical) to be analyzed and resolved. In this section we provide examples of each of these two types of cases. The *exemplars* discussed herein are based on the work of Smith and her colleagues in the United States (Stein, Smith, Henningsen, & Silver, 2000; Smith, Silver, & Stein, 2005a, 2005b, 2005c) and the *problem situations* are based on the work of Markovits and her colleague in Israel (Markovits & Even, 1999a; Markovits, 2003). These two bodies of work were selected for closer examination because they serve to highlight the two broad classes of cases and provide an interesting contrast to the question “What is a case?”

Exemplars: Highlighting Key Ideas about Mathematics Teaching and Learning

The narrative cases created by Smith and her colleagues are *exemplars* or what Shulman (1996) would call *paradigm* cases. That is, they instantiate a broader, more general set of ideas about teaching and learning and are intended to concretize complex practices. These cases draw on data, frameworks, and empirical findings from QUASAR (Quantitative Understanding: Amplifying Students Achievement and Reasoning), a national project in the United States aimed at improving mathematics instruction for students attending middle schools in economically disadvantaged communities in ways that emphasized thinking, reasoning, problem solving, and the communication of mathematical ideas (Silver, Smith, & Nelson, 1995; Silver & Stein, 1996). The project sought to both support instructional improvement efforts in local settings and to carefully document and study classroom instruction and student learning outcomes.

Three keys ideas about mathematics teaching and learning emerged from this research and provide the core set of ideas that are exemplified in the cases: 1) cognitively challenging mathematical tasks provide the greatest opportunities for students to develop the capacity to think and reason; 2) the cognitive demands of tasks can (and frequently do) change during a lesson (i.e., a task that starts out as challenging might be transformed during instruction to a less rigorous exercise);

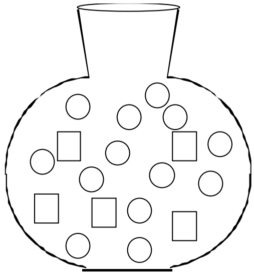
and 3) a teacher's actions and interactions with students during classroom instruction are crucial in determining the extent to which students were able to maintain a high level of intellectual engagement with challenging mathematical tasks (Stein, Grover, & Henningsen, 1996; Henningsen & Stein, 1997).

To date, Smith and her colleagues have written four casebooks (Stein et al., 2000; Smith et al., 2005a, 2005b, 2005c) each of which contains 4-6 rich narratives that take you into one of the QUASAR middle school classrooms and serve to exemplify key features of instruction associated with the implementation of challenging mathematical tasks. The materials in these volumes are intended for use with both prospective and practicing teachers at all levels K-12. The casebooks can be used as the foundation for courses or individual cases can be used in other teachers' professional development programs.

Characteristics of the Exemplars

Specifically, each case-episode portrays the events that unfold in an urban middle school classroom as the teacher engages his or her students in solving a cognitively demanding task that has the potential to engage students in high level thinking about important mathematical ideas (Stein, Grover, & Henningsen, 1996; Stein & Smith, 1998; Smith & Stein, 1998; Stein et al., 2000). For example, Marie Hanson (the teacher featured in one of the cases in Smith et al., 2005a) and her students explore several candy jar problems that involve constructing equivalent ratios of discrete objects and finding the missing value in proportions when one of the quantities is not given (see Figure 2). Since students in Ms. Hanson's class have not previously learned a procedure for solving missing-value problems, they must invent their own strategies rather than applying memorized rules that have no meaning to them.

Each case begins with a description of the teacher, students, and the urban middle school, so as to provide a context for understanding and interpreting the portrayed episode. It then goes on to describe the teacher's goals for the lesson and the unfolding of the actual lesson in a fairly detailed way. Each case depicts a classroom in which a culture has been established over time by the implicit and explicit actions and interactions of a teacher and her students. Within this culture, a set of norms have been established regarding the ways in which students are expected to work on a task (e.g., being willing to take risks, being respectful toward members of the classroom community, being accountable for exampling a solution method).



1. This jar contains Jolly Ranchers (the rectangles) and Jawbreakers (the circles).
   a. What is the ratio of Jolly Ranchers to Jawbreakers in the candy jar?
   b. Write as many ratios as you can that are equivalent to the first ratio that you wrote down.
2. Suppose you have a larger candy jar with the same ratio of Jolly Ranchers to Jawbreakers (5 to 13), but it contained 100 Jolly Ranchers? How many Jawbreakers would you have?

Figure 2. Candy Jar problems from the Case of Marie Hanson.

The cases illustrate authentic practice – what really happens in a mathematics classroom when teachers endeavor to teach mathematics in ways that challenge students to think, reason, and problem solve. As such they are not intended to be exemplars of best practice to be emulated but rather examples to be analyzed so as to better understanding the relationship between teaching and learning. In some case-episodes, like the Case of Marie Hanson, the cognitive demands of the task are maintained as the task is actually enacted; in such cases, various classroom-based factors can be identified which support students' high-level engagement with the task (e.g., pressing students to provide explanations, modeling high-level performance, scaffolding students' thinking). For example, in the Case of Marie Hanson the teacher consistently requests that her students provide explanations for their solutions and ideas and often allows students to use the overhead projector at the front of the room in order to do so publicly. In addition, the teacher also asks students to further explain ideas presented by others (e.g., "I asked if anyone could come to the front of the room and help April put the finishing touches on what was the start of a well-thought out strategy", "I asked the class if anyone had any questions about Jerry's work so far"). By asking students to do the explaining and sense-making she is sending the implicit message that she believes they are capable of such work and that she expects them to do it.

In other case-episodes, the cognitive demands of the task decline as the task is enacted; in these cases, other classroom-based factors – factors that serve to inhibit students' ability to engage with the task at the intended level – can be identified (e.g., the teacher specifies the exact procedures or steps that students need to follow). As such, the cases highlight a set of classroom-based factors that can support or inhibit student engagement with important mathematical ideas in an effort to help teachers develop a reflective and analytical stance towards their own practice (see Henningsen & Stein, 1997).

In addition to highlighting a set of pedagogical moves (i.e., factors) that support (or inhibit) student learning, each of the three more recent casebooks (Smith et al., 2005a, 2005b, 2005c) also highlight a key set of mathematical ideas in a particular domain (i.e., rational numbers and proportionality, algebra, geometry and measurement) and explicitly call attention to the ways in which pedagogy supports or inhibits students' learning of specific mathematical ideas.

Take, for example, the case mentioned earlier in which Marie Hanson and her students are working on the candy jar problems (Smith et al., 2005a). Ms. Hanson selected the problems to build on students' earlier experiences with ratios and has sequenced the problems in a way that helps students to develop increasingly sophisticated proportional reasoning strategies. As students work on the tasks individually and in small groups, the teacher closely monitors their work, noting the approaches they are using and the difficulties they are having. When it is time to share solutions with the entire class, the teacher selects students to present specific strategies. During the whole class discussion the teacher presses students to provide explanations, encourages communication between students, and asks questions that provide opportunities for students to make conceptual connections and analyze mathematical relationships. In the following excerpt from the case we see students trying to make sense of the incorrect strategy for problem 2 that has been presented by Jordan – "since you have to add 95 to get to 100 Jolly Ranchers, I did the same thing to the Jawbreakers – I added 95, so the answer is 108."

> *Does everyone agree with Jordan?" I asked. Jerry volunteered that he did not because "the problem specifically said that the new candy jar had the same ratio of Jolly Ranchers to Jawbreakers and that in Jordan's new jar the ratio was almost one to one." He went on to say he didn't know why Jordan's way was wrong, but he was sure that he was. Still unfazed, Jordon asked Jerry how he did the problem. Jerry came up to the overhead and said that he went back to his old strategy of 1 Jolly Rancher to 2.6 Jawbreakers. He wrote his solution while he explained: "If one Jolly Rancher turns into 100 Jolly Ranchers, it must have been multiplied by 100. And so, the 2.6 Jawbreakers also have to be multiplied by 100." He put the pen down and appeared to reflect for a moment. "I guess I did the same thing to both numbers, too. But I multiplied. You added," he concluded.*
>
> *At this point, I invited the class as a whole to reflect on Jerry's new jar versus Jordan's new jar, "Which jar has the same ratio of Jolly Ranchers to Jawbreakers as our first jar?"* (Smith et al., 2005, p. 34)

There are several noteworthy aspects to this excerpt. First, the teacher (Marie Hanson) has selected a student to present a strategy that is incorrect and she invites the class to take a position – do they agree with Jordan or not. Second, Jerry and

Jordan seem more than capable of carrying on a discussion without any intervention from Ms. Hanson. Finally, Ms. Hanson poses a question that builds on Jerry's observation regarding the new ratio and challenges the class to examine the two solutions using these criteria.

The Case Discussion

It is important to note that learning from cases is not self-enacting. Reading a case does not ensure that the reader automatically will engage with all the embedded ideas or spontaneously will make connections to their own practice. In order to learn from cases, teachers must engage in two types of processes: analysis and generalization. Analysis involves the careful examination of the case, focusing on the teacher's decisions and interactions with students in light of the goals that she wants to accomplish with respect to student learning. Generalization involves viewing the particularities of case-based episodes as instantiations of a broader set of ideas about mathematics, about teaching, and about learning.

Through our work with cases, we have found that readers of a case need to engage in specific activities related to the case in order to maximize the opportunities for learning. These activities include: solving and discussing the mathematical task on which the case is based, reading the case guided by a framing question, engaging in small and whole group discussions of the case centered on the framing question, and generalizing beyond the case to one's own teaching practice and to a larger set of ideas about mathematics teaching and learning. Each of these activities will be discussed in more detail in the sections that follow.

Solving the mathematical task. Having teachers complete and reflect on ways of solving the task that is at the heart of the case is critical to a rich and successful case discussion for several reasons. First, teachers go on to read the case with much more interest and confidence if they first engaged with the mathematical ideas themselves. Also, engaging in the tasks allows misconceptions that the teachers themselves may have to surface. Second, teachers are "primed for" and able to recognize many of the solution strategies put forth by students in the case. This way, teachers' understanding of the multiple pathways to solving the problems becomes strengthened and their perception of student thinking becomes sharper. Finally, Steele (2008) argues that working on the task and then engaging in a discussion of the case provides teachers with the opportunity to integrate their knowledge of mathematics with their knowledge of pedagogy and "create a more powerful learning experience than either activity might have afforded individually" (p. 15). Towards this end it is critical that teachers engage in a mathematic discussion that serves to support *their* learning.

In solving the second candy jar problem shown in Figure 2, teachers generally use one of four strategies: (1) factor-of-change (i.e., since there are 20 times as many Jawbreakers in the new candy jar, there must be 20 times as many Jolly Ranchers); (2) scaling-up (i.e., generate ratios equivalent to 5:13 such as 10:26, 100:260 using a ratio table or some other means); (3) unit rate (i.e., the strategy used by Jerry in the excerpt previously presented; and (4) cross-multiplication. During a discussion of the task, the facilitator presses teachers to compare the various strategies and discuss the ways in which different strategies are related. While this appears to be a simple task, it often uncovers misconceptions teachers have regarding the nature of multiplicative relationships (referred to earlier) and reveals their limited understanding of the cross multiplication algorithm. Research shows that teachers have difficulty differentiating situations in which the comparison between quantities is multiplicative rather than additive, tend to use additive strategies when multiplicative approaches would be appropriate, and do not recognize ratios as a multiplicative comparison (Post, Behr, & Lesh, 1988; Simon & Blume, 1994; Sowder et al., 1998). Hence problems like the Candy Jar task can provide an opportunity to closely examine the relationships between quantities and to consider approaches that would help one make sense of the situation (Boston, Smith, & Hillen, 2003).

Reading the case. Reading the case with a framing question in mind appears to lead to more “active” reading of the case and more thoughtful and focused participation in the case discussion. The framing question is intended to highlight what each case can best contribute to teachers’ investigation of teaching and learning. These questions generally focus on identifying key pedagogical moves made by the teacher during the lesson and the impact these moves have on students learning of mathematics. The key here is not to focus solely on what the teacher is doing, but rather how the teachers’ actions and interactions impact students’ opportunities to engage in and learn mathematics. For example, as teachers read the Case of Marie Hanson, we ask them to identify ways in which her pedagogy appeared to support students’ learning of mathematics throughout the lesson. Teachers are likely to generate a list that includes (but is not limited to) some of the following ideas: the teacher gained knowledge about concepts and how students think about them prior to the engaging students in the lesson; the teacher asked an open-ended question; the teacher was flexible – she did not decide on exactly what the homework would be until she determined what students would be ready for; the teacher monitored students’ understanding; the teacher selected students in a particular way; the teacher used the “wrong” answer as an opportunity to expose the fallacy of an approach; and the teacher chose students wisely (she knows her students).

Discussing the case. The case discussion is intended to help participants analyze the mathematical and pedagogical ideas in the case. The question that framed the initial reading of the case can serve as a starting point for discussion. Facilitating a case discussion itself is skill that teacher educators need to learn. A complaint sometimes raised about case discussions is that they can be “all over the board”, with the facilitator appearing to have only loose control over what gets talked about and how. Facilitators must listen intently to the participants and learn how to steer the conversation in useful directions. Toward this end, it is important for the facilitator to have specific learning goals in mind for the case discussion.

With respect to the case of Marie Hanson, a facilitator may want teachers to be able to learn how to interpret and selectively use various student-generated strategies (such as those presented by Jordan and Jerry) in whole-class discussions. To accomplish this goal, the facilitator would not only pose tasks that provided opportunities for teachers to notice how Marie Hanson used students responses, but would also listen carefully during the case discussion to the teachers' analyses of what occurred in Marie Hanson's class. A good facilitator would highlight and reinforce those comments that related to Marie Hanson's productive use of student responses, would acknowledge but not extend or elaborate those comments that took the discussion in different directions, and would summarize across relevant comments in order to emphasize the points that could be taken away from the case related to her goal.

Consider, for example, a discussion of the Case of Marie Hanson that took place in a content-focused methods course (Stein, Engle, Hughes, & Smith, 2003). The teachers in the course consisted of four practicing teachers working on their master's degrees (two certified for elementary teaching and two for secondary mathematics), and thirteen prospective teachers serving as full-time teaching interns while completing a Master of Arts in Teaching program (twelve seeking elementary certification, and one seeking secondary mathematics certification). During class, the teachers were asked to consider the following question: "What did Marie Hanson do to support student learning?" They were asked to back up their claims with evidence from the case in the form of paragraph numbers, which had been marked on the case itself.

Through this discussion, the facilitator wanted to draw teachers' attention to a set of practices that Marie Hanson employed in order to orchestrate a whole-class discussion in which the mathematics of the lesson was made salient to the students. In particular the facilitator wanted teachers to identify five specific practices in which Ms. Hanson engaged that supported her students learning: 1) anticipating student responses to the bag of marbles task, including the incorrect additive strategy; 2) monitoring students' work on and engagement with the tasks and noting who had used what methods; 3) selecting particular students to present their mathematical work such as Jordan; 4) sequencing the student responses that will be displayed in a specific order, such as beginning the second problem with an incorrect strategy; and 5) helping the class make connections between different students' strategies used and between the responses and key mathematical ideas.

Teachers discussed the question in small groups for approximately 22 minutes, followed by a 34-minute whole-class discussion, which was videotaped and transcribed. Each of the five practices was discussed at least once in the course of this discussion and at least 5 of the 17 items that came up in the discussion were specifically relevant to one or more of the target practices. The discussion of the practices accounted for about 33% of discussion time. While the facilitator recorded all the contributions made by teachers during the discussion, she probed and questioned more on the items that were directly related to her goals for the lesson.

Generalizing beyond the case. Following the analysis of each case, teachers are invited to engage in one or more activities in which the mathematical and pedagogical ideas discussed in the case are connected to their own practice or to other related ideas and issues regarding mathematic teaching and learning. According to Stein and her colleagues (Stein et al., 2000, p. 34):

> In order to "grab hold" of classroom events, to learn from examples, and to transfer what has been learned in one event to learning in similar events, teachers must learn to recognize events as instances of something larger and more generalizable. Only then can they accumulate; only then will lessons learned in one setting suggest appropriate avenues for actions in another.

This often begins immediately following a case discussion by having teachers consider the lessons learned from the case that applies to teaching more broadly. This helps teachers see particular events that occurred during the analysis of a specific case as instantiations of something more general. In addition, there are three types of connections to practice in which we ask teachers to engage: enacting high-level tasks in their own classrooms, analyzing their own teaching, and working on specific issues that were raised for them during the case reading and analysis. These activities are intended to invite exploration of and critical reflection on a teacher's own practice.

Returning to the example presented in the last section in which teachers in a content-focused methods course had the opportunity to learn about five practices for orchestrating a productive mathematics discussion, we consider whether or not teachers were able to generalize the ideas beyond the case. In an assignment given near the end of the term, participants in the course were asked to analyze the similarities and differences between the teaching of the two case teachers (of the four that had been analyzed during the course), to consider the implications of these differences for student learning, and then to discuss the lessons they had learned for their own teaching from doing this. More than half of the teachers in the course selected Marie Hanson as one of the teachers to be analyzed and made many statements in the "lessons learned" section to make it clear that they found the practices relevant not just to the case study teacher, but also to what they planned to do in their own classrooms. Thus it appeared that participants had begun the process of appropriating the practices from the case study teacher to their own instruction.

The Outcome: What Teachers Learn

An important question to consider is what teachers learn from their participation in these case discussions and related activities. Given the amount of time it requires to prepare for and conduct the activities described herein, the outcome needs to match the investment of resources. While our work in this area is ongoing, several studies we conducted provide evidence of teacher learning from engagement in cases and related activities. Specifically, the research indicated that the cases we have created have helped teachers develop a deeper understanding of mathematics content (e.g., Henningsen, 2008; Steel, 2008), a broader range of pedagogical practices (e.g., Hillen & Hughes, 2008; Stein, Engle, Hughes, & Smith, 2003), an understanding of the relationship between teaching and learning (e.g., Smith, 2003) and the capacity to reflect on and learn from the analysis of practice (e.g., Henningsen, 2008).

For example, in the study of teacher learning of the five practices described earlier, Stein and her colleagues (Stein et al., 2003) concluded that: (a) course participants recognized the five practices as a valuable aspect of what they had learned from the course; (b) course participants could learn to identify the five practices when they were embedded in various teaching contexts; and (c) some of the course participants had begun learning to apply the five practices to new pedagogical situations. Hence these studies provide evidence that teachers can learn important aspects of *mathematics knowledge for teaching* (Ball, Thames, & Phelps, 2005) through the analysis of cases and related activities. A key question that we are currently exploring is how teacher learning impacts their instructional practices and, ultimately, the learning outcomes of their students. Towards this end additional research is needed.

Problem Situations: Highlighting the Dilemmas of Practice

The cases created by Markovits and Even are what have been referred to as problem situations**.** The first cases were created for workshops with practicing junior high school teachers (Even & Markovits, 1991, 1993) and focused on the concept of function. Research on the use of these cases indicated that they raised teachers' awareness of students' ways of thinking (Even & Markovits, 1993). This was followed by the development of cases for workshops with practicing elementary school teachers (Markovits & Even, 1994, 1999a, 1999b). In order to enable teacher educators to use the cases in a more systematic way, a casebook was written (Markovits, 2003) for elementary school teachers. The book includes the cases, the analysis of each case (from the mathematical point of view and from the didactical point of view) and suggestions for the facilitator on how to work with the cases. The cases are now being used in courses for prospective teachers and in courses and workshops for practicing teachers.

Characteristics of Problem Situations

Mathematics Classroom Situations (MCS) are cases that focus on specific problematic teaching situations (Barnett, 1998). They are characterized as classroom situations involving mathematics, in which a problem, a dilemma, a debate, or some tension is involved (Markovits & Even, 1999a). They may be real events that took place in a classroom, or hypothetical situations, based on students' ways of thinking and conceptions as identified by research and personal experience. In each situation the teacher is invited to respond to a student's hypothesis, question or idea. The MCS are relatively short and do not include detailed background data, leaving it to the teachers' imagination to complete the background details according to their own experience. These cases invite teachers to suggest ways of responding to the student, based on their understanding of the student's thinking. Examples of MCS are shown in Figure 3 (Markovits, 2003).

All of the situations encourage teachers to elaborate pedagogical content issues such as: what does the student understand or not understand, how the teacher should respond to the student and which response will be the more efficient. Some of the situations also raise mathematical issues that might be challenging for or unclear to teachers. Thus the focus on such situations is also on mathematical content knowledge. For example, in the Division with Remainder situation (shown in Figure 3) teachers might argue that the student is correct and the equal sign should be used since the two answers are the same. Others, although understanding that the situation shows an incorrect placement of the equal sign, might suggest that the student is correct if he is in the lower grades of elementary school since he does not know how to divide without stating the remainder. The Height situation focuses on beliefs about mathematics as well, and also raises issues such as: the connection between mathematics and everyday life, the possibility of having two different answers--one in mathematics and one in everyday life, and the integration of everyday life into school mathematics.

<u>The Missing Number</u>

A student was asked to solve the following exercise:

$$35 - [\quad] = 12$$

The student told the teacher: I will do 35 + 12 because you have here "minus", and I need to use the reverse operation, so I use the "plus."

How would you respond?

<u>Division with Remainder</u>

A student was asked to fill in the correct sign < or = or >, in the following:

$$59 \div 42 \square 359 \div 342$$

The student said he will write the equal sign because*:*

"59 ÷ 42 = 1 remainder 17

359 ÷ 342 = 1 remainder 17

So in both exercises the answer is 1 and the remainder is 17, and that's why they are equal."

How would you respond?

<u>Height</u>

A student was asked the following question:

"The height of a 10 year old boy is 1.5 m. What do you think his height will be when he is 20?" The student answered: "In mathematics it will be 3 meters, because 1.5 × 2 = 3, and in everyday life it will be about 1.80 meters."

How would you respond?

Figure 3. Examples of core versions of Mathematics Classroom Situations.

The fact that MCS are relatively short might be seen as a disadvantage, since unlike the exemplars discussed earlier, no context information is provided. In MCS there is no information about the grade level of the student who produced the response, there is no description of what happened prior to the situation, there is no further explanation about what the student says or does, and there is no further conversation between the student and the teacher or between the student in the situation and other students in class. But all of these missing factors might be seen as an advantage, allowing teachers to think about the situation according to their own knowledge and experience. Teachers might argue about the grade level of the student and prepare different responses for students of different ages. They might suggest different scenarios which could lead to the present situation. They need to be very careful in analyzing the student's idea or question in order to understand what the student meant. The fact that the situation does not describe any action taken by the student's teacher forces the teachers to develop their own reactions. Thus, MCS put teachers in a situation in which they have to struggle with both pedagogical and mathematical issues, in which they need to be creative, to think in different directions, to try and understand what is "going on in the student's head" and to come up with several possible responses.

The MCS also have an extended form that includes responses that are presented as if they were given by other teachers. Some of these responses were actually suggested by teachers during MCS discussions while others were written by the author in order to highlight various issues with which teachers were intended to grapple. The teachers are asked to react to these "other teacher responses." Figure 4 presents responses that accompany the Division with Remainder situation in its extended form.

The first purpose of such responses is to enrich and expand the issues raised by the core form of MCS. Usually some of the responses raise issues limited to the situation presented, while others raise more general issues regarding mathematics teaching and learning. In the example of the Division with Remainder Situation (Figure 4) each of the first three teacher responses introduces a different possible mathematical mistake. The fourth response raises a dilemma: is the answer right if the student is in the third grade, but incorrect if he is in the sixth grade? In other words, does the correctness of the answer depend on the student's age? The fifth response is given by a teacher who knows the mathematics involved but does not understand what the student does not understand. The teacher in the last response puts the finger on the exact difficulty expressed by the student and uses small numbers and pizzas to explain the meaning of the remainder. In order to focus participants' reaction to the "other teacher responses" the following criteria are used (Markovits & Even, 1999a):

– Is there any problem regarding mathematical content knowledge?
– Does the teacher understand what the student does not understand?
– Does the teacher's response concentrate on the student's misconception?
– Does the teacher's response emphasize rituals? Does it pertain to meaning?
– Is the response teacher-centred? Student centred?

1. We will accept the answer. The student was asked to fill in the correct sign. He was not asked to explain but he added a reasonable explanation. Maybe if he would think in a different way, he could find the answer without using calculations. He could notice the numbers:

   59 42
   359 342

   The same number (3) was added to the divisor and the dividend. So we get the same answer.
2. The answer you have is correct, but you should solve the exercise like this:

   $$\overset{1.17}{59 \div 42} = \overset{1.17}{359 \div 342}$$
3. The student is wrong. I will explain that the answer to this problem is:

   $59 \div 42 = 1\frac{17}{59}$, $359 \div 342 = 1\frac{17}{59}$, because we divide the remainder as well.
4. A student in the second or third grade can fill in the equal sign, but a student in a higher grade should know that the division of 17 into 42 is not the same as the division of 17 into 342.
5. Before we check if the answer is correct or not I will ask the student and the whole class: Which one is bigger $\frac{1}{2}$ or $\frac{1}{4}$ and why. When the student will claim that $\frac{1}{2} > \frac{1}{4}$ I will ask which one is bigger and why: $\frac{10}{40}$ or $\frac{10}{20}$? The students will answer: When the nominators are equal the fraction with the smaller denominator is bigger. Now I will ask: Which is bigger $\frac{17}{342}$ or $\frac{17}{42}$? The answer will be, of course, $\frac{17}{42} > \frac{17}{342}$. The conclusion is that $59 \div 42 > 359 \div 342$ and thus the student's response is incorrect.
6. In my opinion the student does not understand the meaning of the remainder. Even if the student is in the third grade, it is important to work with him on the meaning of the remainder but with examples which involve small numbers. For example: $3 \div 2 = 1$ and remainder 1, $5 \div 4 = 1$ and remainder 1. I will talk with him about pizzas. If I have 3 pizzas and I want to divide them between 2 children or I have 5 pizzas which I want to divide between 4 children. Will the children in both cases eat the same amount of pizza? In both cases each child will get one whole pizza but in the first case each child will get another one half while in the second case only one quarter. I think that even a kindergarten child can understand this.

Figure 4. Extended form of the Division with Remainder situation.

Not all teacher responses can be analyzed according to all criteria. For each situation some criteria are more relevant than others and some additional criteria may be taken into consideration.

The second purpose for the use of "other teacher responses" is to involve in the discussion participants who might not feel comfortable in suggesting their own response. Some participants may be concerned that their own response to the situation might turn out to be a not very good one, and even may be criticized by other participants. Some might find among the "other teacher responses" the response they were thinking about but hesitated to say out loud. Thus the analysis of an "other" response may actually be an analysis of *their own* response which is done without the identification of the participant.

Work with MCS

MCS can be used in a variety of ways (e.g., as the focus of a one or two semester course, during a short workshop, integrated into didactic courses) with diverse groups of teachers (e.g., practicing teachers, teachers preparing to become mentors at the elementary level, prospective teachers in their last year of study, mixed groups of prospective and practicing teachers, groups of elementary school teachers and junior high school teachers as part of a program devoted to the transition from elementary school to junior high school). In addition, MCS can also be used during mathematics lessons in elementary school classrooms. The teacher presents the core form of the situation to her students saying that "a student from another class was given …" and invites the students to react (Markovits, 2008). The purpose of using MCS with students is to confront them with the mathematical dilemma in the situation and to encourage them to explain the mathematics involved and justify their answers.

The core version of MCS together with the "other teacher responses" provides a variety of options in working with teachers. Teachers can be asked to respond to the situation in a number of different ways, each of which serves a different purpose:

Individual response – participants can be asked to respond individually (either orally or in writing). In this way, each participant has to decide how she or he would react before listening to other opinions. The individual reaction might then serve as a basis for the whole group discussion.

Small group discussion – participants can be asked to react during a small group discussion that enables the exchange of opinions among a small number of participants. In such settings, participants can disclose their thoughts without hesitation since they will be heard only by a few peers. During small group discussions participants can be asked to try and come up with one reaction that reflects the opinion of all or of most of the group members. Next, one member from each group can present the small group's opinion to the whole group, or several opinions if the members did not agree. Also the arguments can be shared.

Whole group discussion – during whole group discussions a variety of ideas are made public, thus providing the opportunity for the facilitator to move the discussion in different directions. But not all participants can (due to time) or will (due to disposition) explicitly express their viewpoints. Thus, some participants might become passive, waiting for others to express their opinion rather than offering to share their perspectives. Whole group discussions are not a threat to the participants who might hesitate to express their ideas. They can listen to others and compare the ideas expressed during the discussion to their own reactions.

Work with MCS can start with individual responses, move to small group discussions, and proceed to a whole group discussion. Following the whole group discussion, individuals could again be asked for their individual opinion. The individual opinions given prior to and following the whole group discussion could then be compared. Alternatively, a facilitator can start with whole group discussion, continue the discussion in small groups and then conclude by asking for individual opinion. The analysis of "other teacher responses" can be done individually, in small groups or as a whole group, either after the participants have given their individual opinion or after the discussion. Again, the participants can compare their individual opinion before and after the analysis of the "other teacher responses."

The time devoted to one MCS may also vary. Facilitators can deal with two or three different situations in one meeting or devote the whole period of time to only one situation. Of course this decision will dictate the depth at which the situation will be treated. When several situations are presented during one meeting they may have something in common (e.g., they all might focus on the number zero) or they may deal with different mathematical topics and pedagogical aspects. The form and depth of work with MCS is mainly a factor of the situation itself, the participants, and the facilitator.

Example from Working with MCS

The following example is drawn from a whole group discussion on the Height Situation (see Figure 3) that took place during a semester course focused on MCS. The participants were 20 elementary school teachers who participated in a two-year program designed to prepare mentors at the elementary level. The discussion was held during one of the first meetings. Prior to the discussion the participants were asked to indicate in writing how they would respond to the student who produced the solution presented in the MCS.

The discussion began by having teachers report on what they had written individually. Some teachers suggested that the student featured in the MCS was correct in giving two answers and thus separating mathematics and everyday life. Other teachers suggested that the first part of the student's response ("in mathematics it will be 3 m") was incorrect.

T1: I think the student gave a good answer.

T2: The answer was given by a bright child. He did not start from 0. He knows that the growth is not linear.

T3: He explains, but I would not like him to fall into this mistake. Many students see two numbers, so they multiply, divide, add or subtract.

T4: I am teaching both math and science. Although the thinking in the first part of the answer might be correct I would not accept it because not everything we do in math exists in life.

T2: I think the child is trying to act clever. He knows this [the first part] is nonsense.

T5: I think he gave a wonderful answer. He knows how to separate mathematics and reality. He gave a beautiful answer.

Class: Many calls of yes.

Although the participants were asked to consider how they would respond to the student, most of them focused on whether the student was right or wrong and continued to struggle with this issue without actually addressing the question posed. Even when T8 suggested weight as another context, T9 went back to the issue of whether the answer is correct or not as shown in the following excerpt:

T6: I was a little disturbed by the separation he did. Like there is no connection between mathematics and reality. I liked the answer of T2.

T5: What I meant was that he was able to separate between mathematics and everyday life in this case.

T7: I am thinking of estimation here. Estimate the height of people around you.

Class: He did it already.

T8: I would start like T2. But I would use weight and start with a baby born which is about 3 or 4 kilograms, and after one year his weight is growing about 3 times. And what will happen after two years? Then I will go back to heights.

T9: … I think that the student has made a mathematical mistake here, by saying that it is 1.5x2=3, because nobody said that he grew at the same rate. So it is incorrect to say in mathematics it is 1.5x2=3. It is right but not with these data.

At this point, the facilitator tried to move the discussion toward the initial question that had been posed by asking teachers: “So what would you say to the student?” T9 indicated that she would say “that the ratio is not constant.” This lead to a discussion of whether or not such a problem should be used during mathematics lessons since it could “mislead” students as shown in the following exchange:

T10: The child might think the teacher is trying to mislead him. She gave twice the age, so what does she want us to do: to multiply by two.

T11: It is like forcing the mathematics on something incorrect.

T4: The child thinks “You tried to mislead me but look this is not true in everyday life.”

T11: There are so many situations in which we can use mathematics. I would not give a problem like this.

T12: When a teacher gives a problem like this as part of mathematics lesson she has a purpose. If she gives such a problem and knows that the answer to it is that she shouldn't give such a problem, what exactly is her purpose?

T6: To check if the children separate or do not separate mathematics from everyday life.

T12: Why is this in a mathematics lesson?

T13: I think that almost all of us, if we were in class we would tell the student that he gave a good answer, he separated mathematics and everyday life, I mean the student is thinking. To come now and say why you multiplied by 2, maybe now we teachers think about this, in class there is no time to think so I would say to him: Good for you! You know how to separate, it did not look reasonable to you so you used estimation, it's good you did not relate to the given numbers only.

The teachers continued to discuss the issue of the two answers given by the student. The facilitator did not interfere, but stopped the discussion after a few more minutes and presented a new situation. In the next meeting "other teacher responses" for the Height situation were presented and analyzed and the discussion was opened again.

The comment of T13 sheds light on the importance of cases in teacher education, suggesting that in class, in real time, there is not enough time to think about and come up with good answers to students' questions or ideas. Cases provide an opportunity for teachers to take the time to think deeply about a situation, to discuss a situation with other colleagues, to react to "other teacher responses" and to rethink the situation from several angles. By going through this procedure with many and different MCS, teachers can enrich their repertoire of situations and become more ready to deal with MCS when they occur in real time in their classrooms. Moreover, they become aware of the potential of such situations to enhance learning, and start looking for rich MCS in their own classrooms by turning students' questions or ideas into Mathematic Classroom Situations. The characteristics of a good MCS are captured in the reflection of a teacher at the end of course:

> A good Mathematics Classroom Situation is a situation which questions the teacher's self confidence just for a moment or two .… no matter if what the student said is right or wrong. A good situation raises a dilemma, an uncertainty, and a need for discussion. The teacher has to think about the way he would explain to the students why this was so and so. The explanation cannot be immediate and just be pulled out as can be done with routine issues in which the answer is immediate and trivial. A good Mathematics Classroom Situation is not forgotten at the end of the meeting but it keeps rolling in the head and almost always is driven by a desire to discuss it with other teachers.

The Outcome: What Teachers Learn

Our research with practicing teachers showed that MCS enhanced teachers' mathematical content knowledge and pedagogical content knowledge, encouraged them to rethink their teaching, sharpened their attention to students' explanation and focused teachers' responses on what the student did not understand (Even & Markovits, 1991, 1993; Markovits & Even, 1994, 1993a, 1993b). The following are some excerpts of elementary teachers (prospective and practicing) who were asked "What is the most important thing you take from the MCS course?"

> "I learned that it is not enough to know the mathematics. The teacher has to understand the difficulties students have and to be able to supply an immediate answer. Also the didactical courses should be changed by bringing in the class reality, same as we did here."

> "As a teacher I have to be aware of the way students think and to try and find out what exactly he was thinking in a given situation so that I know how to respond. Also, I need a deep knowledge of mathematics so I can relate to the mistakes made by the student and to the correct mathematical facts."

> "The teacher has to be very professional. He has to know a number of explanations for the same material, thus if the student has difficulties in understanding one explanation, the teacher can explain in a different way."

> "I learned to look at the student at a different angle, to try and understand him and his way of doing mathematics."

> "Following the situations we had in class I started to pay attention to situation that 'happen' to me when I am teaching in class. I started to pay more attention to the ways students solve problems, what I have to do in class, and how can I encourage students to reach the correct solution by themselves without my explanation."

DISCUSSION

Shulman (1986) argued that to call something a case is to make a theoretical claim – that is, that any story that is called a case must be a case *of* something. In this chapter we have attempted to explicate the essential attributes of a case and to explore two different visions of what the foci of a case can be – complex teaching practices (in the case of exemplars) and the problematic aspects of performance (in the case of problem situations). These two types of cases make salient the range of ways in which complex classroom events can be used in teacher education settings as contexts for developing the knowledge and skills needed to respond to the complexities and demands of real-time teaching.

A key issue in selecting a case is not the length, the context, or even the foci. Rather, it is the extent to which the material can engage teachers in analyzing authentic problems of practice that will help to build their capacity to make sound judgments in the classroom that matters most. Toward that end, the success of a case depends in large measure on the skill of the facilitator in highlighting the question, "What is this a case of?," thus stimulating learners "to move up and down, back and forth, between the memorable particularities of cases and the powerful generalizations and simplifications of principles and theories" (Shulman, 1996). Like an experienced teacher, a facilitator must decide "when to let students struggle to make sense of an idea or problem …, when to ask leading questions, or when to tell students something" (NCTM, 1991, p. 38). The choices made by the facilitator have an influence on the direction of the discussion, on the depth and range of issues that are brought to the fore, and on the opportunities participants have to gain new insights, question current practices, and to continue to learn and develop as professionals.

Awareness of the vital role facilitation plays in the case method is evidenced by the growing availability of both specific and general materials for facilitators. For example, many of the casebooks described earlier (e.g., Smith et al., 2005a, 2005b, 2005c; Markovits, 2003) provide extensive materials for facilitators (e.g., identification and analysis of key mathematical and pedagogical ideas in a case, questions to stimulate discussion and reflection, sample responses from previous participants) that are intended to help them to use the specific case materials productively. More general advice for case facilitators is available from the experts from Harvard Business School (Barnes, Christensen, & Hansen, 1994) in a book aimed at helping facilitators "learn more about the skills and knowledge essential to case method teaching" and a seminar program "to help case method instructors become more adept in their craft" (p. 1).

Cases, regardless of how carefully they are selected or how skillfully they are facilitated, are not a panacea for all the shortcomings of teacher education. We must be aware of the potential pitfalls as well as the promise of these materials. Ball (2001) cautions us to remember that the analysis of cases (and other artifacts of practice) is intended to help teachers in their ability to make instructional decisions in the classroom, not to help teachers become more skillful at performing analysis for its own sake. Avoiding this pitfall requires keeping the work of teaching as a focus and making connections between the task at hand (analysis of a case) and the real work that teachers do (teaching children in classrooms).

CONCLUSION

Although only twenty years have passed since Shulman proposed "case knowledge" as a component of "teacher knowledge," and less than fifteen years have passed since the first mathematics cases were published, much progress has been made in integrating case methods into teacher education. Of course much more needs to be done. First, more cases are needed. Most of the cases currently available were developed for teachers in the elementary and middle grades. Few cases currently exist that focus on mathematics teaching in the early grades (PreK-2) or on high school mathematics. Teachers at all levels need opportunities to develop their knowledge base for teaching through critique, inquiry and investigation into the work of teaching--a task for which cases are particularly well-suited.

Second, we need to continue to design teacher education experiences that make use of case-based methods and other pedagogies and study what teachers learn from these experiences. According to Sykes and Byrd (1992), the selection and sequencing of cases with other elements of teacher education is a complex curricular issue. Ball and Cohen (1999) caution us to design professional education experiences so as to avoid "simply reproducing the kind of fragmented, unfocused, and superficial work that already characterizes professional development" (p. 29).

Finally, Griffin (1999) states that research on case use "is fragmented and fragile" echoing a concern expressed by many in the field that "enthusiasm for case use in teacher education comes primarily from advocacy" (p. 138). Hence additional research is needed to further explore issues of teacher learning (e.g., what do teachers learn from different types of cases and how they learn it) and how what teachers learn impacts their teaching performance.

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