Beliefs and Attitudes in Mathematics Education

New Research Results

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During the last fifteen years, research on affect has been of considerable interest to the mathematics education community. Researchers with an interest in mathematics and gender had a look at aspects of affect in their research studies right from the beginning. Similarly many studies of mathematical problem solving had a growing interest in affect. The main focus of research are now student beliefs and teacher beliefs which are identified as important factors for those influencing learning and teaching.

The thirteen chapters of this book involve many aspect of research on affect like theoretical problems of defining beliefs, the complex relationship between content knowledge and affect, espoused beliefs and teaching practice, domain-specific beliefs as well as the relationship between special learning conditions and affective reactions.

This book is addressed to all researchers with an interest in affect, emotions, beliefs and teaching mathematics. It gives an orientation and an overview to many aspects of research in the field.
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FOREWORD

In a survey paper on affect in mathematics education, McLeod writes:

Although affect is a central concern of students and teachers, research on affect in mathematics education continues to reside on the periphery of the field (McLeod 1992, p. 575).

In the last fifteen years, we have seen a growing interest in affect in mathematics education. For example, research on gender aspects in mathematics learning has led, in particular, to interest in attitudes towards mathematics (Fennema and Sherman, 1976; Forgasz and Leder, 1996). Affect has come to be seen as having an important influence on mathematics learning: it is a factor in the observed differences in performance between girls and boys.

A second, more interesting, direction in affect research has involved studies of mathematical problem solving (e.g., McLeod and Adams, 1989). Observations of students carrying out various problem-solving tasks demonstrated that their reaction during the solution process could not be understood as a purely cognitive process. One consequence of these studies was the emergence of a new category of affect: emotions. Mandler’s concept of the construction of emotions is used as psychological basis theory (Mandler, 1989). McLeod (1992) sees beliefs, attitudes and emotions as categories of affect, and intensity and stability as characteristics that distinguish these categories. Goldin (2002) extended the categories of affect by defining the category of “values, ethics and morals” and described the mental representation of all affective categories. Research into affect has always been accompanied by a discussion of the definition of these categories (Hart, 1989; Furighetti and Pehkonen, 2002; Törner, 2002; Di Martino and Zan, 2001, 2008; McLeod and McLeod, 2002). Despite this body of work, Sfard writes:

Finally, the self-sustained “essences” implied in reifying terms such as knowledge, beliefs, and attitudes constitute rather shaky ground for either empirical research or pedagogical practices – a fact of which neither research nor teachers seem fully aware (Sfard, 2008; p. 56).

Common to all research into affect is the idea that the categories of affect are based on mental systems, and that these mental systems have a crucial influence on all the processes of students’ mathematics learning and teachers’ mathematics teaching. Hence there has been much research into students’ beliefs and teachers’ beliefs (Leder, Pehkonen and Törner, 2002), as well as into processes to change beliefs and attitudes. A further important question pertains to the origin of affect (Fennema, 1989; Evans, 2000; Schlöglmann, 2008). The consensus is that beliefs, attitudes and values are the consequence of an evolutionary process that involves all of an individual’s experiences with mathematics throughout their entire life. An exception is emotions, which are based on an individual’s general mental mechanisms, evoked when reacting to situational and local problems. Nevertheless, these reactions can also lead to longer-term consequences (Goldin, 2000).
The emergence of empirical methods was of great importance in the history of research into affect in mathematics learning and mathematics teaching. It began with studies of the relationship between attitudes towards mathematics and the performance of students. These studies used attitude scales to measure attitudes (Fennema and Sherman, 1976; McLeod, 1994). In later years, a number of researchers developed methods for investigating beliefs and attitudes (Leder and Forgasz, 2002, 2004). Of particular interest was research into self-concepts (Bandura, 1997, Malmivuori, 2001).

In the evolution of research into affect one may discern a number of distinct currents. Influential in continental Europe was the paper of Pehkonen and Törner (1996), as well as the MAVI conferences that brought together, initially, researchers from Finland (Pehkonen, Hannula and Malmivuori) and Germany (Törner and his collaborators), and later also from Cyprus (Philippou and his collaborators) and Italy (Furinghetti, Zan and their collaborators) – see the MAVI Proceedings. Other currents evolved in Great Britain (Evans, Lerman and Tsatsaroni) and Belgium (Op’t Eynde, De Corte and Verschaffel).

The genesis of the present book was a MAVI conference in St. Wolfgang, Austria, in 2008. This event continued the tradition of MAVI conferences in bringing together researchers for the purpose of presenting new results in the field of affect, and to provide an opportunity for intensive discussion of these results. Thus this book contains contributions to many aspects of the field. Below we present a short overview of the variety of themes discussed in this book.

The book starts with a discussion of theoretical aspects of the field, focussing on problems relating to the definition of beliefs and to specifying the status of beliefs within the affective domain. The complex relationship between subject matter knowledge and pedagogical content knowledge is presented in a case study of a kindergarten teacher. During the teaching process, teachers must convey their beliefs in an intension of practice. This leads to a complex relationship between espoused beliefs and teaching practice.

In some cases, the construct “beliefs in relation to mathematics” is too general; beliefs within mathematical domains such as algebra, geometry and statistics and probability should then be distinguished. By considering domain-specific beliefs we can provide explanations for teachers’ responses to education reform processes.

The idea of distinguishing beliefs within specific areas can be taken further. For example, it is important to gather information about teachers’ beliefs in relation to the central concepts that may be found in school mathematics curricula. Thus the present book contains chapters devoted to teacher beliefs in relation to proofs, problem solving and percentage learning.

In an effort to improve the quality of school education, concepts such as competency-based and outcomes-based education – new buzzwords in education debates today – have entered educational theory. However, new concepts and buzzwords are only one side of the story: the transfer of concepts in teaching practice is still often a difficult problem. Therefore a number of chapters in the present volume explore, directly or indirectly, impediments to concept transfer.

In mathematics education, mathematics efficacy beliefs are seen as a predictor of successful teaching; therefore it is important that the beliefs held by prospective
teachers with respect to the efficacy of mathematics be compared to those of practicing teachers.

Prospective teachers undertaking university education bring with them beliefs and attitudes towards teaching acquired during their years as students at school. These beliefs can be a barrier to developing new teaching competencies – we should therefore find out more about these beliefs.

Studying teacher beliefs is an important prerequisite to changing and improving teaching at school; nonetheless, we ought to keep in mind that students also develop beliefs and that these beliefs have a crucial influence on their mathematics learning. It is argued that finding out more about student beliefs, gender aspects, as well as the effects of age, can help improve mathematics education.

It is noted that students are accustomed to solving overly simplified mathematical tasks that are specified by question texts which contain exactly the right amount of data, and which do not contain irrelevant information. To improve student problem-solving competency, it is important to study what happens when problems are not as clear-cut.

Different countries have different classroom practices for introducing mathematical concepts. Investigations of the influence of classroom practices in which textbooks are used reveal interesting consequences for learner identity.

Finally, we would like to thank all the individuals, and institutions, who contributed to the production of this book. We thank the authors, who contributed far more than their own papers: through their commentary and discussion both during the conference and during the review process, they helped in the development of all the other papers as well.

We sincerely thank the publisher, Peter de Liefde, for his valuable support.

The “Bundesinstitut für Erwachsenenbildung (bifeb) in St. Wolfgang am Wolfgangsee” provided a beautiful conference venue – a photograph of which adorns the front cover of this book – together with excellent facilities, and its staff were most friendly and helpful.

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FOREWORD


1. BELIEFS – NO LONGER A HIDDEN VARIABLE IN MATHEMATICAL TEACHING AND LEARNING PROCESSES

In this paper, we discuss some theoretical issues raised in research on beliefs in mathematics learning and teaching. These include struggles around the issue of defining beliefs, for which we suggest that constitutive elements of a structural framework for characterizing beliefs may provide a useful way forward. We consider the role played by beliefs in some influential approaches to mathematics teaching and learning: problem solving, change and development, and sense-making. We elaborate on the affective domain in relation to mathematical beliefs, highlighting positive possibilities that can be associated with affect and beliefs. Our concluding comments mention briefly some important research directions.

REVISITING BELIEFS RESEARCH

Much research has been conducted on the essential role of beliefs in learning and teaching mathematics (cf. Thompson, 1992; Richardson, 1996; Philipp, 2007). The book edited by Leder, Pehkonen and Törner (2002), which had its origins in a conference at the Mathematics Research Institute in Oberwolfach, Germany in 1999, sought to address mathematics teachers’ beliefs, students’ beliefs, beliefs about the self, and some rather general beliefs related to the domain of mathematics and mathematics learning. Thus it made sense to identify beliefs with respect to their various holders. How diverse beliefs are mutually related then becomes an interesting issue for exploration.

Researchers on beliefs apply the term to a number of different notions. In his beliefs-alphabet, Mason (2004) quite creatively lists related concepts, alluding to the different themes that arise in discussions of beliefs. With respect to the relationship between knowledge and beliefs, Boaler (2001) stresses that “for many years educational theories have been based upon the assumption that knowledge is a relatively stable, individual characteristic that people develop and carry with them, transferring from place to place” (p. 3). Törner (2001) analyzes students’ ad hoc answers to specific mathematical questions, and concludes that the mental net of “knowledge” is dominated by beliefs. In particular, he argues that beliefs serve as nodes in this net if there is no grounded knowledge available – i.e., so-called “knowledge structures” are primarily belief structures.

Lerman (2001) identifies two major strands of research concerning beliefs: analysis and classification of beliefs, and monitoring changes in beliefs over time.

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GOLDIN, RÖSKEN AND TÖRNER

Rokeach (1968) had already contributed fundamental work on the structure of belief systems, which he believed were organized psychologically but not necessarily logically. Likewise Green (1971) refers to structural aspects of beliefs, considering some beliefs to be more central than others – and, consequently more difficult to change (cf. Törner & Pehkonen, 1996). Cooney (2001) emphasizes that belief structure is crucial, since information about how beliefs are formed can potentially help us understand how they change.

A few studies use well-established categorizations of beliefs in order to document changes in individuals’ beliefs about the nature of mathematics and its teaching and learning (e.g. Liljedahl, Rolka & Rösken, 2007a). Recently conducted research draws on earlier, fundamental work by Green (1971), seeking to identify structural features in terms of dimensions (Pehkonen, 1995; De Corte, Op ’t Eynde & Verschaffel, 2002; Rösken, Hannula, Pehkonen, Kaasila & Laine, 2007). Changes in beliefs have been discussed in terms of conceptual change by a number of authors (Murphy & Mason, 2006; Pehkonen, 2006; Liljedahl, Rolka & Rösken, 2007b), extending a mostly cognitively-grounded theory to the concept of beliefs.

Today, nine years after the Oberwolfach conference, the discussion has become more elaborated and advanced; but the classical questions on identifying and characterizing beliefs in mathematics education are still debated (Leder, 2007). More and more articles consider beliefs, even when that is not their central focus. Many conferences have what might be called “belief sessions,” with a wide variety of contributions.

Our objective in this paper is to discuss some relevant theoretical issues raised in the research to this point. In Sec. 2, we engage with some theoretical struggles around the issue of defining beliefs. Then we consider how beliefs interact with three influential approaches to mathematics teaching and learning – problem solving, change and development, and sense-making. In Sec. 3, we consider the affective domain in relation to beliefs, returning in a rather different way to the issue of structure. We conclude in Sec. 4 with a brief comment on methodological issues.

THEORETICAL STRUGGLES

This section revisits the discussion of beliefs with a focus on some theoretical aspects: definitions of beliefs, a constitutive framework that can guide the discussion of beliefs, and how beliefs interact with different approaches to the characterization of mathematics teaching and learning.

Definitions

One thing that has not changed in the last few years is the absence of universal acceptance by mathematics education researchers of a definition of beliefs that can ground the various theories.

Some researchers advocate strongly for the importance of definitions and careful distinctions – for example, Goldin (2002) defines beliefs as “multiply-
encoded, internal cognitive/affective configurations, to which the holder attributes truth value of some kind (e.g., empirical truth, validity, or applicability)” (p. 59), and distinguishes among beliefs, warranted beliefs, and knowledge. Others maintain that the absence of consensus around definitions is not necessarily counterproductive, since beliefs constitute a very flexible and accommodating construct. We do not foresee the situation changing soon (Furinghetti & Pehkonen, 2002; Törner, 2002).

However, we wish to suggest here that introducing constitutive elements of a structural framework guiding our understanding of beliefs may permit an approach different from that of adopting a definition, or at least compatible with several possible definitions, and allow us to represent explicitly cognitive and affective aspects of beliefs that are, to varying degrees, relevant.

Such an approach, in a sense, parallels a development in the history of geometry. Hilbert (1899/1962) made it clear that in an axiomatic approach to geometry, the old “definitions” of point and line provided by Euclid did not truly serve as definitions, but only delineated possible perceptions for the readers when speaking or thinking about those objects. Likewise, constitutive elements of a framework for discussing beliefs may accommodate distinct perceptions of them.

Törner (2002), for instance, distinguishes the following different aspects, on which we elaborate here:

**Ontological aspects:** Beliefs are always attached to objects of belief. To address a belief, one has to identify the corresponding belief object, for instance, “the philosophy of mathematics” or “the role of the integral.”

The term “object” is well established in attitude theory (e.g. Ruffell, Mason & Allen, 1998) and we would like to encourage its standard use in belief theory as well. The objects of belief can be domain specific, and can be personal, social, or epistemological in nature. There seems to be little restriction: nearly anything may serve as an object with which beliefs are associated. In discussing beliefs, the relevant objects should be named explicitly! When the objects of belief are not made explicit, the discussion sometimes may be characterized as slippery (cf. Hofer & Pintrich, 1997).

**Enumerative aspects:** Beliefs can be regarded as aggregates of mental states. Belief objects can be assigned a (subjective) content set of various possible perceptions, characteristics, suppositions, philosophies, and/or ideologies, which are often simply referred to as beliefs or better, belief states.

Such a content set may contain one, few, or many elements (beliefs), depending on the respective object and the bearer and the time of discussion. With respect to a specific belief object under discussion, the bearer may at any time reduce or extend the set of belief states.

**Normative aspects:** Beliefs are highly individualized. The content set can be modeled as akin to the mathematical notion of a fuzzy set, which means that the elements of the content set possess different weights that are attributed to various perceptions or assumptions. This membership function may be regarded as a measure of the level of consciousness and certitude of the belief bearer, or the degree of activation of the belief.
As Rokeach (1968) and Green (1971) have pointed out, belief states can be distinguished by logical, psychological, epistemological or social aspects. In our terminology, this means that elements in a specific belief set are not interchangeable.

**Affective aspects:** Last but not least, beliefs are interwoven with affect—emotional feelings, attitudes, and values. Thus elements of the content set carry an affective dimension, including some kind of *evaluation measure* that expresses (for instance) the degree of emotional approval or disapproval, favor or disfavor associated with the belief.

As researchers in the mathematical sciences, we have observed few if any beliefs with which the bearer associates no affective loading. Even the task of multiplying 7 by 8 reveals considerable affect when an elementary school child arrives at her belief that the answer is 56, while research mathematicians are typically emotionally invested in beliefs about the truth of mathematical theorems or the validity of proofs. We discuss affective aspects of beliefs further in Sec. 3 below.

Beliefs are highly subjective, and vary according to the different bearers. Thus observers of a specific situation may refer to quite different beliefs. The constitutive framework described above is applicable to individual, “discrete” belief objects, or to complex ones comprising various facets (e.g., domains). It can include subjective individual theories or convictions about central characteristics related to a specific object. It can address beliefs about individual persons, one’s own self, or human beings in general; beliefs about a domain of interest, such as the natural or social environment; or epistemological convictions about the process of gaining insight and knowledge (cf. Köller et al., 2000).

Our goal is to be able to apply the flexible construct of beliefs to various situations pertaining to mathematics education. We would like to identify and understand the positive as well as the negative influences of beliefs—beliefs that can serve as *affordances* as well as those that present barriers. Important questions in understanding the role ascribed to beliefs in different research contexts include: What are the most prominent thematic ways in which the construct of belief is used? At what scale is the construct applied—e.g., is the discussion on micro, meso, or macro level? What functional purposes does the belief construct fulfill? Is it possible to model some of the observed phenomena by assigning functional roles to beliefs?

Mathematics Teaching and Learning Processes

Some definitions or interpretations of beliefs, with implications for their role in mathematics teaching and learning, can be understood by exploring the psychological and/or epistemological consequences of the metaphors or analogies used to describe them. For example, Abelson (1986) provocatively develops the idea that *beliefs are like possessions*. He writes:

> Going further, let us explore several semantic aspects of property possessions: One obtains things, one keeps them, one values them and sometimes loses
them. [...] For the obtaining aspect, there are expressions such as ‘to acquire a belief,’ ‘to inherit the view,’ and so on, as though beliefs were things that figured in some sort of social or physical transfer process. For the keeping aspect, one commonly encounters the wording that someone ‘holds a belief,’ or ‘holds onto a belief.’ Reference is made to valuing by such expressions as ‘to cherish a belief’ and ‘I’m reappraising my position.’ The losing aspect is especially rich in metaphorical phrasings, including ‘to lose your belief in ...,’ ‘to abandon your belief,’ ‘to surrender your principles,’ and so on. Each of these categories gives some support to the ‘possession’ metaphor. (p. 230)

Abelson’s elaborated metaphorical analysis was taken over by Rolka (2006) in her ongoing research. With the possibility of such overarching metaphors in mind, let us next explore some of the roles of beliefs in various approaches to mathematics teaching and learning.

Schoenfeld (1991), Hiebert et al. (1997), and others point out that beliefs about the “vast objects” learning mathematics and teaching mathematics should be explicitly addressed. We shall consider three perspectives that have been and continue to be widely influential – learning as problem solving, learning as change and development, and learning as sense-making.

**Problem solving approaches to mathematics teaching and learning.** Halmos (1980) sees problem solving as at the heart of mathematics (a phrase often quoted). Likewise, Rav (1999) stresses that the “essence of mathematics resides in inventing methods, tools, strategies and concepts for problem solving” (p. 6). Mathematical problem solving can be interpreted narrowly, as specific mathematical activity directed toward a goal, where the solver does not initially know how to reach that goal. But in a much wider sense, problem solving can be understood as a comprehensive approach to mathematics teaching and learning – embodying views like those of Halmos and Rav of what mathematics is, views of what it means to learn mathematics, and how mathematics can be effectively taught. Schoenfeld (2005) uses the phrase, “problem solving from cradle to grave” in his title, to highlight the 20-year history of problem solving approaches to mathematics education. Authors discussing the so-called theory-practice dichotomy in teacher education likewise refer to this broad conception of problem solving (Malara & Zan, 2002).


Such reforms depend to a large extent on institutional reform: changes in the overall mathematics curriculum. They depend even more essentially on
individual teachers changing their approaches to the teaching of mathematics. Teaching reforms cannot take place unless teachers’ deeply held beliefs about mathematics and its teaching and learning change. (p. 99)

But we have also seen, perhaps, the “grave” of problem-solving based reforms to the mathematics curriculum in the United States. One reason offered for their failure is the inappropriate beliefs of teachers (cf. Schoenfeld, 1985, 1994; Frank, 1990; Garofalo, 1989). Such beliefs are often characterised as misconceptions or obstacles, with the primary goal of reformers being to change them. Today, a research project in China is devoted to investigating whether teachers possess adequate beliefs before implementing a new curriculum (Qian, 2008).

In fact, there are numerous papers describing the rather negative influences of beliefs that are incompatible with making problem solving central to the teaching and learning of mathematics. While we do not disagree with these findings, we want to argue for the importance of research that identifies and characterizes constructive, helpful, powerful beliefs. Are there not also many situations where beliefs are guiding and inspiring, and serve as affordances to the improvement of mathematics teaching and learning?

If we return to George Polya’s (1957) main steps in problem solving – understanding the problem, making a plan, carrying out the plan and looking back on what you have done – all of these, as well as the far more specific heuristic processes described in his later books (Polya, 1962/1965) are enhanced essentially by the problem solver’s personal beliefs and speculations regarding possible solutions or solution paths, as well as beliefs in the solver’s own capacity for success. Beliefs can be like “pulsars” generating flashes of intuition, or lighting the way to further and deeper insights. As Fishbein (1987) points out:

For a very long time, reasoning has been studied mainly in terms of propositional networks governed by logical rules. Consequently, the instructional process, especially in science and in mathematics, has tended to provide the learner with a certain amount of information (principles, laws, theorems, formula) and to develop methods of formal reasoning adapted to the respective domain. [...] What has been shown in this work is that, beyond the dynamics of the conceptual network, there is a world of stabilized expectations and beliefs which deeply influence the reception and the use of mathematical and scientific knowledge. (p. 206)

The dynamics of mathematical reasoning – and generally, of every kind of scientific reasoning – include various psychological components like beliefs and expectations, pictorial prompts, analogies and paradigms. These are not mere residuals of more primitive forms of reasoning. They are genuinely productive, active ingredients of every type of reasoning. (p. 212)

Still more broadly speaking, on both micro and macro levels, a continuous process of decision-making is essential for successful problem solving (Schoenfeld, 2005; Malara & Zan, 2002). Malara (2008) also reminds us that “teachers cannot be simple knowledge conveyors” but must assume the role of decision makers (p. 1).
Schoenfeld (2005) describes decision-making as a function of knowledge, goals, and beliefs:

In brief outline: an individual’s beliefs, in interaction with the context, shape the formation and prioritization of goals. Given a particular constellation of goals, the individual looks for and implements knowledge that is consistent with his or her belief systems and is designed to satisfy one or more high-priority goals. As goals are satisfied (or not), or as the context changes, new goals take on high priority, and actions are then taken in the pursuit of these goals. (p. 41)

While stressing that research should focus “on the things and ways that teachers believe” (p. 307), Pajares (1992) refers to “[...] the assumption that beliefs are the best indicators of the decisions individuals make throughout their lives (Bandura, 1986; Dewey, 1933; Nisbett & Ross, 1980; Rokeach, 1968), an assumption that can be traced to human beings’ earliest philosophical contemplations” (p. 307). Obviously, beliefs are both “indicators of decisions” and variables that ultimately influence them. Powerful and effective decision-making, then, is likely to depend in a positive way on the individual’s beliefs.

Törner, Rolka, Rösken and Sriraman (2008) further analyze the decision-making process using Schoenfeld’s (2005) terms. They suggest that goals and beliefs can often be regarded as mutually dependent concepts. In articulating a goal, a teacher expresses implicitly what he or she believes; and conversely, beliefs presuppose and incorporate goals (see also Aguirre & Speer, 2000).

This observation calls attention to another role of beliefs in problem solving contexts – that they can partly serve as forecasting tools or predictive tools. To understand a person’s beliefs enables one to “forecast” to some degree the person’s decisions (e.g., in teaching or learning situations), although the nature of the correlation between beliefs and behavior is itself very complex. As Bandura (1986) states,

People regulate their level and distribution of effort in accordance with the effects their actions to have. As a result, their behavior is better predicted from their beliefs than from the actual consequences of their actions.

While some psychological learning theories focus on the consequences of changes in behavior (e.g., rewards associated with problem solving success), Bandura reminds us of the relevance of the antecedent input (e.g., the beliefs that underly the decisions and choices).

To summarize, while beliefs may be defined in different ways or left undefined, they are fundamental to the discussion of problem solving approaches in mathematics education. They are necessary components in the psychology of how mathematical problems are solved. They are antecedent variables influencing the pedagogical decision-making of teachers. Prevailing beliefs have been perceived as impediments to problem-solving based reforms of the mathematics curriculum and of classroom teaching methods. And most important, there is a need for research on
the positive roles that beliefs play; on how effective beliefs serve as affordances in mathematical teaching and learning.

**Change and Development Approaches to Mathematics Teaching and Learning**

Another set of approaches to mathematics education draw on the idea that a central objective of teaching is to *change and to develop the state of the learner*. Learning itself is seen as fundamentally a process of change in internal mental states. To become accommodated to new insights or new perspectives is one of the challenges posed in a learning situation. These processes then come into conflict with beliefs, since in this context beliefs (like attitudes) may be relatively stable and resistant to change. As in the discussion of problem solving approaches, the conflict occurs both at the level of detailed teaching and learning activities, and at the level of teachers’ roles in facilitating learning.

The *possession metaphor* is just one way to understand resistance to change in beliefs. Many beliefs are linked to the self-concept of the bearer, so that such beliefs function to serve a kind of self-assertion which protects the bearer against uncomfortable ideas. Thus changing one’s beliefs is not normally the first option chosen. In the context of change and development approaches to learning mathematics, assessment becomes an important tool – yet when a student’s assessment suggests deficiency, why should the student (or the student’s teacher) abandon the belief that success in mathematics is dependent on innate talent? We return to this widespread belief in Sec. 3 below.

Green (1971) argues that beliefs never occur as single entities, but appear in clusters. Then one cannot simply replace a single belief (metaphorically speaking), but must exchange sets or bunches of beliefs. Beliefs are reciprocally stabilizing, as they are interwoven into systems with other beliefs (see also Goldin, 2002).

The implication for didactical processes is that beliefs are like inertia (see Pehkonen & Törner, 1996). The role of inertia in daily life is – at first glance – neutral, and so is the role of beliefs in teaching and learning. As long as beliefs are adequate approximations to cognitive structures, the vagueness of beliefs is not problematic. As long as beliefs are organized as mental states open for development, beliefs can be extended without changing them in their core. For example, the (narrow) belief that mathematics is basically arithmetic, and consists just of computation and calculation, is extendable to a similar but more comprehensive belief about algebra and geometry. In classroom activities, some teachers may encourage this point of view, while others may introduce algebra as a new mathematical subject disconnected from arithmetic and its associated beliefs.

There is no universal pattern, but it is clear in the context of change and development approaches to mathematics education that modifying a curriculum entails far more than replacing the syllabi or the textbooks.

In the literature, there are discussions of changing teacher beliefs through *socialization and experience*, change processes that are mostly attributed to teacher education programs (e.g. Richardson, 1996, pp. 110ff). But Nespor (1987) argues
that “beliefs are basically unchanging, and when they change, it not argument or reason that alters them but rather a ‘conversion or gestalt shift’” (p. 321). Pajares (1992) elaborates further:

“When metaphysical and epistemological beliefs are deep and strong, an individual is more likely to assimilate new information than to accommodate it. Posner et al. (1982) suggested that individuals must be dissatisfied with existing beliefs and that new beliefs must be intelligible and appear plausible before most accommodation can take place. Moreover, new beliefs must be consistent with other conceptions in the ecology. Rokeach (1968) would suggest that they must have functional connections to other beliefs in the structure. Learning and inquiry are dependent on prior beliefs that not only make current phenomena intelligible but also organize and define new information. Beliefs are unlikely to be replaced unless they prove unsatisfactory, and they are unlike to prove unsatisfactory unless they are challenged and one is unable to assimilate them into existing conceptions. When this happens, an anomaly occurs – something that should have been assimilable is resisted. Even then, belief change is the last alternative. Posner et al. (1982) found that students in their study rejected new information, considered it irrelevant, compartmentalized their conceptions to prevent it from conflicting with existing beliefs, or even forcefully assimilated it in the face of conflicting logic, reason, and observation before they would consider accommodation.” (p. 320ff)

**Sense-making Approaches to Mathematics Teaching and Learning**

Finally, we consider approaches to mathematics education in which learning is seen as essentially making sense, generating meaning, and/or acquiring or constructing understanding.

Clearly sense and meaning are of vital importance in mathematics. René Thom (1973), for example, writes:

One has not, I believe, extracted from Hilbert’s axiomatics the true lesson to be found there; it is this: one accedes to absolute rigour only by eliminating meaning; absolute rigour is only possible in, and by, such destitution of meaning. But if one must choose between rigour and meaning, I shall unhesitatingly choose the latter. […] The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of ‘meaning’, of the ‘existence’ of mathematical objects. (p. 202).

This issue was very important in connection with the “new math” movement of the 1950s and 1960s, where rigor tended to overshadow the processes of giving meaning, making sense, and building up subjective conceptualizations of mathematical objects. The process of sense making and the genesis of beliefs go hand in hand – the learner searching for sense and meaning develops beliefs about “small objects” (the mathematical objects being studied), as well as beliefs about “larger objects” (e.g., the role of meaning in mathematics).
Correspondingly, we quote the famous mathematician Toeplitz (1927):

All these subjects of infinitesimal calculus are taught today as canonized requisites, the mean value theorem, the Taylor series, the concept of convergence, the definite integral, especially the differential quotient itself, without nowhere touching the questions, Why is that? How does one reach it? All these requisites must haven been objects of a fascinating search, an exciting action, namely at that time they were created. (p. 92ff)

Beliefs, often framed by norms (e.g., Yackel & Rasmussen, 2002), can serve as basic modules for the perception and interpretation of virtual entities. Since an individual’s perceptual and information processing capacity is limited, beliefs help to reduce and structure information to fit constrained patterns. In short, beliefs are at the heart of meanings. Metaphorically speaking, they are loaded with epistemological information (e.g. Verschaffel, Greer & De Corte, 2000; Moschkovich, 1992; Litwiller & Bright, 2002).

Sometimes the information reduction is far from acceptable, and then the problem of changing a belief or system of beliefs enters as a potential obstacle to learning. But Schommer (1990) points out the interrelation between epistemological beliefs and mathematical text comprehension, while underlining the role of productive beliefs. Epistemological beliefs may be decisive for student motivation (Hofer, 1999; see also the discussion of affective structures below). Finally, sense-making processes and beliefs cannot be separated from the “self,” and thus reflect aspects of identity – identity as a teacher (e.g., Lin, 2000; Lin & Cooney, 2001; Boaler & Greenwood, 2000), or as a student (e.g., Boaler, 1999, 2001). Thus we remark again that there is nothing intrinsically negative about beliefs in relation to mathematical sense-making. A belief or system of beliefs can function to facilitate or impede mathematical learning, depending on the belief, the object of belief, and the affordances provided by the belief (see also Schraw & Olafson, 2002).

RELATED CONCEPTS IN THE AFFECTIVE DOMAIN

Emotional Feelings, Attitudes, Beliefs and Values

We have seen that beliefs and systems of belief typically involve complex cognitions. One can discuss the beliefs of oneself and others logically, making use of empirical observations and rational thinking, and explore their implications. In reasoning about beliefs one can thus uncover, present and examine evidence for their truth or falsity, and sometimes this can result in changing them accordingly. But we have also seen in the discussion to this point that beliefs (and the stability of beliefs) are often strongly influenced by affective factors; and beliefs themselves may serve affective functions. Thus as we explore the role in mathematics education of beliefs held by students and teachers, it is essential that we give attention to the full affective domain.

McLeod (1989, 1992, 1994) proposes to regard the affective domain in relation to mathematics education as including emotions, attitudes, and beliefs. These are named in increasing order of stability (over time), and with increasingly prominent
cognitive elements – thus emotional feelings may be fleeting and highly affective, while beliefs are more likely to be stable and incorporate specific, detailed cognitions. Expanding on this discussion, DeBellis and Goldin (1997, 1999, 2006) suggest a tetrahedral model that includes a fourth subdomain – that of values, ethics, and morals. They view each of these four subdomains as interacting with the other three. That is, in order to understand the role played by beliefs, and why certain beliefs are so tenaciously held, we must consider the emotional feelings and attitudes that support them, the emotional and attitudinal needs that they serve, and the values with which they are consonant or dissonant.

For example, beliefs may meet emotional needs or provide defenses from pain. When they fulfill such functions, it may not be easy for someone to relinquish them based (merely) on new cognitions. Let us consider the frequently-expressed belief that a person’s mathematical ability is fixed and innate – “in the genes.” A (hypothetical) high school student who does not do well in mathematics may be drawn to such a belief for emotional reasons. The belief that he himself was “born without math ability” may relieve him of personal responsibility for his lack of success. Since he believes his low capability to be fixed and innate, he can safely consider failure in mathematics to be “not my fault” – he can even take pride in the idea that he is “just not a math person.” The belief assuages guilt, alleviates the pain associated with failure, and provides a “good reason” for him to disengage with doing mathematics before emotional feelings of frustration arise.

Attitudes may be considered either as propensities toward certain patterns of behavior, or propensities toward certain kinds emotional feelings in particular domains, e.g. in relation to mathematics. The belief that one’s own mathematical ability is fixed at a low level may thus encourage attitudes that reinforce avoidance behaviors toward mathematics (“mathematics is just not for me”). Furthermore, in this example the belief may be consonant with a personal value system in which success through hard work is considered meritorious. The belief provides an independent reason why hard work in mathematics cannot bear fruit, through no fault of the student himself.

Likewise, for somewhat parallel reasons, a (hypothetical) teacher may be attracted to the belief that each of students has a fixed, innate mathematical ability – ranging from low to high. Such a belief may help to diminish the teacher’s feelings of frustration with those of her students’ whose learning is slow, or diminish her frustration with herself or sense of personal failure for being unable to improve her students’ learning. To acknowledge the possibility of mathematical talent being acquired may not only be contrary to her experience, but may necessitate confronting emotionally painful issues.

We may endeavor to change various mathematical beliefs of students or teachers, in order to open up possibilities of deeper mathematical understandings or higher levels of achievement. But the above considerations suggest that part of the process of changing beliefs must be creating a context in which it is emotionally safe to do so.
META-AFFECT AND BELIEFS

DeBellis and Goldin introduced the term “meta-affect” to describe – in analogy with “metacognition” – affect about affect, affect about cognition about affect, the affective context of affect, and affective monitoring (DeBellis, 1996; DeBellis & Goldin, 1997, 1999, 2006). Meta-affect has the capability of transforming profoundly the experience of affect: an emotional feeling such as fear, usually considered negative, may be experienced with intense pleasure (as in riding a roller coaster, or viewing a scary movie). Feelings in connection with mathematics, such as anxiety or frustration, may be experienced with pleasurable anticipation (as in looking forward to learning something new and challenging, or heightening one’s interest in a challenging problem).

Thus the meta-affective context in which mathematical beliefs are held becomes an important consideration – one must ask not only what emotional feelings may interact with the beliefs, but how the person feels about having those feelings. Changing the meta-affect around frustration, for example – which may mean coming to feel safe with and enjoy moments of impasse during mathematical problem solving – can be important to changing beliefs whose emotional function is to enable the avoidance of frustration. Meta-affect (whether positive or not) tends to contribute to the stability of beliefs and belief systems in the individual (Goldin, 2002).

AFFECTIVE STRUCTURES

Beliefs and values are components of a recently-developed social psychological construct labeled an archetypal affective structure (Goldin, Epstein, & Schorr, 2007). Such structures are recurrent, idealized patterns that appear to occur when children are engaged in social situations with conceptually challenging mathematics – e.g., working in groups in a mathematics classroom. They were characterized in the analysis of videotapes of urban mathematics classrooms, in low-income, predominantly minority communities in the United States (Alston et al., 2007; Epstein et al., 2007). However, they are conjectured to occur universally, in many different contexts (hence the adjective “archetypal”). Examples of such structures include:

– “Don’t Disrespect Me.” This involves the person’s experience of a perceived threat to his or her status, dignity, well-being or safety. The experience can occur when ideas are challenged during a mathematical discussion. Maintaining “face” comes to supersede the mathematical content under discussion.

– “Check This Out.” This involves the individual’s realization that successful engagement with the mathematics can have a payoff (intrinsic or extrinsic), leading to increased (intrinsic) interest in the task itself.

– “Stay Out Of Trouble.” This involves the person’s avoidance of interactions that may lead to conflict or emotional distress, so that aversion to risk comes to supersede the task’s mathematical aspects.

– “It’s Not Fair.” Here the experience of a sense of unfairness within a group problem-solving effort, e.g. with the level of participation by others in the group, leads to a disinvestment in the mathematical ideas in the task.
There are many more examples. Each such structure – described, of course, in an idealized form – is understood to have developed in the individual, but to be activated in social situations. In analogy with cognitive structures, affective structures are regarded as mutually interacting, and more than one may be active at any given time.

The mutually interacting components of an archetypal affective structure are taken to include: (a) a characteristic pattern of behavior in response to social circumstances, culminating in a characteristic behavioral outcome; (b) a characteristic affective pathway; (c) the significations or meanings of the emotional feelings; (d) characteristic “self-talk” in response to, and evoking, emotional feelings; (e) characteristic problem-solving strategies and heuristics; (f) interactions with the individual’s beliefs and values; (g) interactions with the individual’s self-identity, integrity, and intimacy; (h) meta-affect; and (i) external expressions or manifestations of affect. Structures may “branch” into each other, depending on aspects of the social context in which they are evoked.

In this analysis, mathematical beliefs and values not only have structure and belong to systems of beliefs and values, but are embedded in complex structures that are important to understanding students’ motivations and behavioral patterns. If the validity of such a construct, or one like it, holds up in future research, we shall need to consider how changes in belief may contribute – positively or negatively – to the affective structures that govern student engagement with mathematical ideas.

CONCLUDING COMMENTS

We have examined the importance of beliefs in a variety of contexts pertaining to the teaching of learning of mathematics. A recurring theme is that of structure. Beliefs themselves are structured, in that they are attached to objects, have associated content sets, possess normative aspects, and serve affective as well as cognitive functions. Furthermore beliefs are often linked with each other, in structured systems of belief. Finally, beliefs are embedded in complex affective as well as cognitive structures.

This complexity raises intriguing issues regarding the empirical observation or measurement of beliefs. One approach is quantitative, making use of questionnaires or similar structured instruments. Here one seeks to identify dimensions that can structure the world of beliefs for a specific population. One then investigates relations among these dimensions through the analysis of response patterns, and possibly seeks to associate beliefs or patterns of belief with attitudes, with receptivity to various teaching or learning practices, with sociocultural factors, or with other variables.

Another approach is qualitative, integrating beliefs into broader theories of problem solving, affect and motivation, or learning and teaching, and seeking evidence from statements and mathematical behavior (of students, teachers, mathematicians and/or mathematics education specialists) of how beliefs interact with other factors to influence the phenomena under study.
Ultimately, it is hoped that both qualitative and quantitative investigations can provide useful “windows” on phenomena involving beliefs, allowing studies that are increasingly sensitive and sophisticated. These in turn should allow mathematics educators to identify and make use of the (productive) affordances as well as the (impeding) constraints associated with beliefs.

To sum up, beliefs matter. Their influence ranges from the individual mathematical learner and problem solver and the classroom teacher, to the success or failure of massive curricular reform efforts across entire countries. Due to the efforts of many individuals, it is fair to say that beliefs now constitute “a no longer hidden variable” in research on the teaching and learning of mathematics.

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GOLDIN, RÖSKEN AND TÖRNER


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This chapter presents the case of Lali, a kindergarten teacher committed to improving her mathematical work with young children. Lali’s reflective notes on her Subject Matter Knowledge (SMK), Pedagogical Content Knowledge (PCK), emotions and beliefs are described and discussed. Lali initiated several meetings with us (the authors of this chapter) in which the topic of triangles was discussed. After emotional waviness of Lali during the geometry-oriented meetings, her enthusiasm and joy in the profession were accompanied by higher SMK and PCK.

There is growing awareness among mathematics educators of the importance of early childhood mathematics education and of the central role of affect in mathematics learning and teaching. However, “little research …addresses the topic of mathematics teachers’ affect” (Philipp, 2007. p. 258) and the number of studies on early childhood teachers’ mathematically-related affect is minute. Thus, it seems essential to devote more efforts to the study of early childhood teachers’ mathematically-related affect.

In this paper we describe the case of Lali, an early childhood teacher in a poor neighbourhood. The study highlights the complex relationship between Subject Matter Knowledge (SMK), Pedagogical Content Knowledge (PCK) and affect.

INTRODUCTION

The last two decades have seen a remarkable increase in attention to the mathematics education of young children. Recent mathematics education curriculum and standards documents address the mathematical topics and processes that ought to be part of the mathematics education of young children. For instance, in the NCTM’s recent document: *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics – A Quest for Coherence* the pre-K and K are considered together with grades 1 to 8:

Three curriculum focal points are identified and described for each grade level, pre-K–8, along with connections to guide integration of the focal points at that grade level and across grade levels, to form a comprehensive mathematics curriculum. To build students’ strength in the use of mathematical processes, instruction in these content areas should incorporate—
the use of the mathematics to solve problems;
an application of logical reasoning to justify procedures and solutions; and
an involvement in the design and analysis of multiple representations to learn,
make connections among, and communicate about the ideas within and outside
of mathematics.

(NCTM, 2006, p. 10).

These documents also highlight the importance of developing positive emotions
and attitudes towards mathematics. The first paragraph in the executive summary
of the *Principles and standards for school mathematics* (NCTM, 2000), for
instance, reads:

> Developing a solid mathematical foundation from prekindergarten … is
> essential for every child. In these grades, students are building beliefs about
> what mathematics is, about what it means to know and do mathematics, and
> about themselves as mathematics learners. These beliefs influence their
> thinking about, performance in, and attitudes toward mathematics and decisions
> related to studying mathematics in later years.

(NCTM, 2000, p. 98).

Research suggests that the most critical feature of a high-quality educational
environment is a knowledgeable and responsive adult (Bowman, Donovan &
Burns, 2001; Darling-Hammond, 1997). There are consistent evidence that most
teachers of young children have limited knowledge of mathematics and of the
thinking strategies of mathematics in early childhood (Clements, Copple & Hyson,

Research also indicates that geometry is only superficially discussed both in
kindergartens and in elementary schools (Clements, 2003). A main reason for this
neglect is a lack of SMK and PCK of geometry of the teachers (Clements, 2003).
Consequently, even the most basic parts of the geometry curriculum in the
kindergartens, such as the recognition of examples and non-examples of triangles,
is too often reduced to presenting only prototypical examples of triangles (Table 1,
Cell 1) and prototypical examples of squares (Clements & Sarama, 2007).

Accordingly, there are consistent calls for initiating professional development
programs for early childhood teachers that focus on SMK, PCK, and disposition
toward mathematics in general (Sarama & DiBiase, 2004) and toward geometry in
particular (Clements & Sarama, 2007).

In the last years, we have devoted extensive efforts to working with preschool
teachers in low-income areas in Israel.

One of our most prominent observations was that geometry is hardly present in
these kindergartens. In our conversations with the kindergartens teachers many
defined themselves as suffering from geometry anxiety. They also felt insecure
when it came to geometry and were reluctant to deal with geometry in their
kindergartens. In our work we draw on two aspects of Shulman’s dimensions:
SMK and PCK.
In his seminal work, Shulman (1986) introduced an analysis of knowledge components that are necessary for teaching: Subject Matter Knowledge (SMK), Pedagogical Content Knowledge (PCK), and Curricular Knowledge (CK). Regarding the first dimension, SMK, Shulman argued that

…to think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the structures
of the subject matter... [which] include both the substantive and the syntactic structures. The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structures of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established.

(Shulman, 1986, p. 9)

Shulman further explained that teachers’ SMK refers to the amount and organization of knowledge per se in the mind of the teacher”, because “the teacher need not only understand that something is so; the teacher must further understand why it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied.

(Shulman, 1986, p. 9)

Shulman, who suggested the notion PCK, argued that this notion involved a dramatic shift in teacher understanding from being able to comprehend subject matter for themselves, to becoming able to clarify the subject matter in new ways so that it can be grasped by students. According to Shulman (1986) PCK is characterized, as “that special amalgam of content and pedagogy that is uniquely the province of teachers” (p. 8). It is “a second kind of content knowledge… which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching (p. 9). Moreover, PCK includes “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and background bring with them to the learning” (p. 9), and familiarity with research studies “of students’ misconceptions and their influence on subsequent learning” (p. 10).

The terms Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) have contributed greatly to the discussion of what teachers need to know to become professional. Shulman’s perspective has gained considerable attention in mathematics education and an increasing number of studies refine and extend various aspects of this theory (e.g., An, Kulm, & Wu, 2004; Ball & Bass, 2003; Even & Tirosh, 1995; in press; Fennema & Franke, 1992; Graeber, & Tirosh, in press; Grossman, 1990; Hill, Rowan, & Ball, 2005; Hill, Sleep, Lewis, & Ball, 2007; Marks, 1990; Sowder, 2007; Tirosh, & Graeber, 2003; Tsamir, 2006; 2007; in press; Wilson & Wineburg, 1988).

When applied to working with kindergarten teachers on triangles (and on other geometric figures), we found that it is important to clarify that mathematically, the spectrum of examples and non-examples of triangles is infinite and that it is essential to base the differentiation between examples and non-examples of triangles solely on a definition of a triangle. Accordingly, we emphasize that it is important to encourage the children in the kindergarten to classify geometrical figures to triangles and non-triangles not according to their global appearance, but according to the critical properties of triangles. This aim is by no means trivial as young children are often at the most basic level of van Hiele (van Hiele & van
Hiele, 1958; van Hiele, 1999). That is, when asked to determine if a given shape is a triangle, young children often consider only the global appearance of the whole shape and not its properties (Clements & Sarama, 2007). Consequently, some triangles are often classified as non-triangles while some non-triangles are frequently categorized as triangles (Battista, 2007; Clements, 2003; Hannibal, 1999; Tsamir, Tirosh, & Levenson, in press).

Taking the SMK and PCK dimensions into account, we defined four groups of geometrical figures with respect to classifying figures into triangles and not-triangles: “friendly triangles” (i.e., triangles that are easily identified as such, Table 1, Cell 1), for example, the equilateral and isosceles triangles (e.g., Hershkowitz, 1990). “Unfriendly triangles” (i.e., triangles that are often mistakenly identified as non-triangles, Table 1, Cell 2), for example, Burger and Shaughnessy (1985) found that young children did not identify as a triangle a long and narrow triangle, such as a scalene triangle in Cell 2, even when they admitted that the figure had three points and lines. “Friendly non-triangles” are geometrical figures that are easily identified as non-triangles (Table 1, Cell 3), for example, geometrical shapes, such as circles and squares (e.g., Tsamir, Tirosh, & Levenson, in press). “Unfriendly non-triangles” (geometrical figures that are often mistakenly identified as triangles, Table 1, Cell 4), for example, a rounded “triangle” is often identified as a triangle (Hasegawa, 1997).

In our work with the kindergarten teachers we use these distinctions for discussing SMK and PCK issues related to introducing triangles in their kindergartens. Our attempts to promote the kindergarten teachers’ SMK and PCK of triangles are accompanied with explicit discussions on the participants’ feelings, emotions and beliefs.

Research on teachers’ beliefs, emotions, attitudes, and affect has vastly increased during the last two decades. These terms, however, are not used in a uniform way in mathematics education. Philipp (2007) provided several working definitions and descriptions of these terms. He defined affect as “a disposition or tendency or an emotion or feeling attached to an idea or object” (p. 259) and argued that affect is comprised of emotions, attitudes and beliefs. In our work with the kindergarten teachers, we focused mainly on two aspects that Philipp included in affect: Emotions and beliefs.

Emotions are characterized, by Philipp as: “Feelings or states of consciousness, distinguished from cognition. Emotions change more rapidly and are felt more intensively than attitudes and beliefs... Emotions may be positive or negative” (p. 259).

Beliefs are described as “Psychologically held understandings, premises, or propositions about the world that are thought to be true... Beliefs, unlike knowledge, may be held with various degrees of conviction” (p. 259).

A main aim of our work was to create a supportive and caring atmosphere and to encourage the kindergarten teachers to openly discuss their emotions and beliefs about mathematics, about learning and teaching mathematics in kindergartens and about themselves as learners and teachers of mathematics.

Our interactions with the kindergartens teachers were constructed so that three, crucial elements in such interactions were considered: SMK, PCK, and affect. In
some cases, we worked with several teachers, In others, with one teacher. This paper tells the story of our work with Lali, on triangles.

BEFORE THE BEGINNING

Lali’s kindergarten is located in a poor neighbourhood. At that time Lali had nine years of experience. She was a kind, responsible and extremely dedicated teacher. She had neither studied mathematics during her teacher education, nor after graduating from the teacher college.

At the year-school starting on September 2004 Lali had a mixed-aged class with pre-school children aged 4-5 years and 5-6 years. Such mixed-age kindergarten classes are quite prevalent in Israel.

Lali heard us giving a lecture on intuition and mathematics, where we expressed our desire to work with kindergarten teachers. After the presentation she addressed us, noting that she would very much appreciate meeting with us as she “wants to give her children ‘the best’ in mathematics”. We set a meeting, and this was the beginning of our work together.

We met with Lali once a week for about two hours. All sessions were recorded. In these meetings we talked about mathematics (addressing SMK), about children’s conceptions; tasks and activities (addressing PCK) and emotions and beliefs (addressing affect).

At the end of each session Lali answered in writing the (same) reflective questionnaire that included questions about her SMK, her PCK, her work with the children, and her emotions and beliefs (see Figure 1). We regularly visited Lali in her class, and afterwards discussed our observations of her work and the work of the children.

| Date:   __________ |
| 1. Mathematical gain - In your view, did your mathematics improve this week? Explain. How do you feel about it? |
| 2. Psycho-didactical gain - In your view, did your knowledge of “kindergarten mathematics” improve this week? Explain. How do you feel about it? |
| 3. Practical gain - In your view, did your work on mathematics in your kindergarten improve this week? Explain. How do you feel about it? |
| 4. Thoughts ideas, dilemmas - Are there any “extras” you would like to share with us, or ask us? |

Figure 1. Questions addressing Lali’s SMK, PCK, emotions and beliefs
We present here significant instances of Lali’s triangles-related progress as expressed in her written, reflective notes.

OUR TRIANGLE-MEETINGS WITH LALI

At the end of one of our meetings, Lali shared with us an activity and “something interesting” that happened in her class.

Lali: I worked with several children on ‘cut-and-colour triangles’. I have done it many times, children love it...

Dina: Could you please describe this activity?

Lali: It is a good task. It is interesting. I bring a large drawing of a round [circle shape] pizza, that has on it some red colour for the tomato sauce and some yellow colour for the cheese. Four black diameters are marked on the pizza. I ask the children to cut out ‘triangles of pizza’. Then each child gets one triangle of pizza and draws on it mushrooms, olives, whatever, [adds enthusiastically]. It’s a daily activity. It speaks to the children. It makes the triangles relevant to life... It’s really fun. Only that...

Pessia: What??

Lali: You see, Ron, a very clever kid, said that he thinks that a triangle can’t have a bend, because all sides must be straight. What should I do with Ron? I know that we need to finish soon, just give me an idea.

Dina: Well... Ron got a point here. We MUST talk about triangles. Please, don’t work on triangles in the kindergarten until we talk.

In her reflective notes after this session, Lali referred to the triangles incident:

Mathematical gain – I don’t know triangles. I was certain that I know, I was wrong, I-do-not-know. I must learn with you. I’m not sure that I’ll understand.

Feelings – I feel horrible! Shock! I don’t know any more where to trust my knowledge. Not to know triangles - I feel sad and stupid. This is shameful.

Lali became aware of her limited, mathematical knowledge, about triangles and expressed a need to study this topic with us.

Lali’s feelings about this unpleasant revelation were extremely negative: (1) horrible, (2) uncertain – “I don’t know... where to trust my knowledge”; (3) stupid, (4) ashamed, (5) sad.

However, she did not fall to despair. She was determined to use our assistance, to study and improve her knowledge of triangles.

Psycho-didactical gain – I learnt not to use the pizza activity for triangles. Now, I don’t know how to work on triangles in the kindergarten. I don’t know enough about triangles myself. I should know first.
**Feelings** – I feel bad, frustrated and very insecure. For a long time, I believed that I was working well, but I enthusiastically taught something that is apparently an error. I’m embarrassed. I must learn. Not to mislead the kids. I’m not sure if I’ll dare talking about triangles in the kindergarten ever again! I’m not good enough. But, I believe I have to know, in case a child says something. I really want to be excellent for the kids.

Here, Lali expressed more critical, responsible conclusions regarding the ‘pizza activity’, and the need to stop working with it in its present form. She learnt that the pizza activity was probably mathematically incorrect.

Lali described her negative feelings: (1) feel bad, (2) frustrated, (3) insecure and incompetent, “I’m not sure if I’ll dare”, “I’m not good enough” (4) embarrassed.

Her beliefs were that (1) she [the teacher] should know about triangles even if she decides not to teach it anymore, and (2) only if one is knowledgeable of a topic, one should work with the kids on the topic.

**My work in the kindergarten** – I am going to stop working with children on triangles, surely for now, but maybe I’ll never do it again.

**Feelings** – Perhaps, to be on the safe side, I shouldn’t do any geometry in my class. It’s better not to teach than to teach errors. I have no words to describe my awful feeling about the harm I may have done to the kids.

Lali made a practical “not to do” decision. She decided to stop working with the children on triangles, at least temporarily.

Lali’s feelings are so (awfully) negative that she does not find words to describe them. It’s easy to feel her sorrow as well as her great commitment to the children. She expressed once more her beliefs of the need to base teaching on solid knowledge, and her firm view that “not teaching” is better than “teaching errors”.

In the “requests” section Lali expressed her wish to study, with us, about triangles. Consequently, this episode elicited three sessions that were explicitly dedicated to adult-geometry (not to be immediately implemented in the kindergarten).

**The SMK Section: What are Triangles?**

In these meeting we talked about triangles. We discussed various, mathematical definitions, necessary and sufficient attributes, naming mathematical concepts, examples and non-examples, the differences between definitions, properties and theorems. Lali showed great interest in our discussions, and easily formulated a wide range of examples and non-examples of triangles. In her reflective notes on the third week she wrote:

**Mathematical gain** – I learnt so much: Definitions of triangle, mathematical definitions (I’m not certain I understood all of them perfectly). I know the role of examples and non-examples, how to identify them and how to formulate them.
Feelings – I feel great! It feels really good to know; I’m glad to know that now I know. There are always new things to learn, but I should not be afraid. I can learn mathematics.

At this point Lali knew a lot about triangles, and she was happy with her new, geometrical knowledge. She also acknowledged that knowledge is continuously being acquired and that there is always something new to learn in any topic.

“Knowing” made Lali feel extremely positive: (1) “I feel great”, good feeling (2) glad (3) “love” learning, (4) fulfilling, (5) confident - “not be afraid”

Lali believed that she can learn mathematics that she might face more difficulties in the future, but she should and could handle them.

Lali neither related in her writing to PCK gains, nor to her work in the kindergarten, but in the thoughts, ideas and dilemmas section she wrote:

What are good K-triangle-activities? I’m not sure and it frightens me. Before doing anything about triangles with the kids I would like us to talk. Also to look at activities I used to do about triangles and at games that I have in class, to be sure that I use them properly with the children.

Lali felt that she had gained relevant, mathematical knowledge. This made her happy and confident in her mathematical abilities. However, Lali sensitively differentiated between her own mathematical understanding (SMK), and the knowledge about triangles that she needed in order to bring “triangles” to her kindergarten kids (PCK). She asked us to discuss these issues in our next meetings.

The PCK Section: Triangles in the Kindergarten

Three meetings were dedicated to conversations about triangles in the kindergarten. We first addressed the mathematical property of triangularity and the psychodidactical property of “friendliness” (See Table 1). We then drew some conclusions regarding what is likely to be easy (figures in Cell 1 and in Cell 3), and what is likely to be difficult for children (figures in Cell 2 and in Cell 4). Then, we discussed with Lali the importance of promoting the children’s geometrical reasoning via encouraging them to use the critical attributes of the figures and not their global appearance. For this purpose we introduced Lali to the first, second and third levels of the van Hiele theory (e.g., van Hiele & van Hiele, 1958). According to this theory, at the most basic level, students’ use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components. Students at this level can name shapes and distinguish between similar looking shapes. At the second level students begin to notice that different shapes have different attributes but the attributes are not perceived as being related. At the third level, relationships between attributes are perceived. At this level, definitions are meaningful but proofs are as yet not understood.

Then, we suggested a kindergarten’s-triangle-definition that included a non-minimal list of critical properties (“three-ness”, “closeness”, “sides – straight”, “edges – sharp”), and emphasized that we strongly recommend not to use expressions such as “almost triangles” in her class. Our work concluded with
formulating, together, different tasks and classroom activities intended to promote children’s knowledge of triangles. Together we formulated activities that are apt to encourage the use of the kindergarten definition of triangles. At the end of the last PCK meeting, in her reflective notes, Lali related to the psycho-didactical gain, and to the feelings and thoughts, ideas and dilemmas sections. She wrote:

**Psycho-didactical gain** – I learnt to evaluate what children know, I know now some related ways to work with them in whole class settings, in small groups and when needed, individually. I can formulate various triangle-activities, for example, paper drawings, cards, and shapes on the floor. I differentiate between friendly and unfriendly shapes, and I use this information when working with the kids.

**Feelings** – I feel really good; it’s so good to know the adult-mathematics and to be able to use it as a base when deciding how to work with the children. It’s a great responsibility to do geometry in the kindergarten, and now I feel that I can do it. I am able to do it right. I am really glad about it. Now, with this knowledge I should do triangles in the kindergarten.

Lali listed several PCK elements that she acquired in these sessions. For example, how to examine ‘who knows what’; ‘what are easy and what are difficult examples and non-examples;’ ‘Work in different-sized groups’; and ‘to formulate and to analyse triangle-activities’.

Lali’s feelings were very positive. She expressed, in her choice of words, (1) confidence – “I feel that I can do it right”, and (2) glad, to describe her mathematical and didactical self-reliance, and her joy for feeling that way.

She believed that she has the related SMK and PCK, and she can and should work with children on triangles.

**My work in the kindergarten** – I am using all the triangle-activities that we discussed, and I decorated the kindergarten with triangle-mobiles and pictures.

**Feelings** – Great! Just great! I see the children happy when learning such complicated and demanding topics, and I am so proud that they know. It is very important and satisfactory for me. I am happy to be able to do all this. I know about triangles, about kids and about activities, so I can lead responsible triangle-activities. I had many ups and downs since I showed the pizza activity.

Lali declared implementing what she learnt in the kindergarten realm. Our observations confirmed that Lali and the children worked with great enthusiasm on the various, triangles activities, and that Lali exhibited, in her work, solid SMK and PCK.

Lali expressed her feelings using words such as (1) just great, (2) pride, (3) satisfaction, and (4) happiness. These sayings indicate that Lali felt good about her triangle-work in the kindergarten, that she was glad that the children learnt with
“great enjoyment”. At the end of these triangle-meetings, Lali declared once more that teaching should be based on solid SMK and PCK knowledge.

SUMMING UP AND LOOKING AHEAD

This case study accompanied Lali in her “ups and downs” of knowledge, emotions and beliefs. We started with Lali excitedly showing us an example of what she regarded as “a good and interesting” triangle activity. Lali exhibited positive feelings of self confidence, joy and enthusiasm. Yet, these feelings related to an activity in which a non-triangle was mistakenly classified as a triangle. At the end of the described sessions Lali had similar feelings, but at this point her knowledge of triangles and of children learning triangles, remarkably improved. This raises the complex issue of the relationship between knowledge (SMK, PCK) and affects (emotions, beliefs).

In our meetings with Lali, we first discussed her own SMK, asking her, at that point in time, not to discuss triangles in her class. Only then we devoted time to discussing various PCK aspects. The question of the “good” sequencing of SMK and PCK should further be researched.

In this chapter we described one case: that of Lali. A question that naturally rises relates to the generalization that can be drawn from this case. That is, are the affection conclusions for the specific teacher general or just typical for Lali’s case? We view this study as a starting point in a long journey of exploring the interesting interplay between SMK, PCK and emotions in preschools.

NOTES

1 In Hebrew when a slice of such a pizza is called a ‘triangle of pizza’.

REFERENCES


AFFECT, SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE


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