Teacher Knowledge and Practice in Middle Grades Mathematics

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This book presents a coherent collection of research studies on teacher knowledge and its relation to instruction and learning in middle-grades mathematics. The authors provide comprehensive literature reviews on specific components of mathematics knowledge for teaching that have been found to be important for effective instruction. Based on the analysis of video data collected over a six-year project, the chapters present new and accessible research on the learning of fractions, early concepts of algebra, and basic statistics and probability.

The three sections of the book contain chapters that address research on the development of mathematics knowledge for teaching at the undergraduate level, instructional practices of middle-grades teachers, and the implications of teacher knowledge of mathematics for student learning. The chapters are written by members of a research team led by the Editor that has been working for the past six years to develop practical and useful theories and findings on variables that affect teaching and learning of middle grades mathematics.

Mathematics knowledge for teaching is a topic of great current interest. This book is a valuable resource for mathematics education researchers, graduate students, and teacher educators. In addition, professional developers and school district supervisor and curriculum leaders will find the concrete examples of effective teaching strategies useful for teacher workshops.
TEACHER KNOWLEDGE AND PRACTICE IN MIDDLE GRADES MATHEMATICS
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Gerald Kulm

Texas A&M University, USA
To Joan

My loving wife and best friend
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This volume presents original research on teacher knowledge and its relation to instruction and learning middle-grades mathematics. The intent is to describe the specific aspects of teacher content and pedagogical knowledge that are important for effective instruction, as well as examining the instructional practices that predict gains in student learning. The chapters are written by members of a research team led by the Editor that has been working for the past six years to develop practical and useful theories and findings on variables that affect teaching and learning of middle grades mathematics. Focusing on the learning of fractions and early concepts of algebra, the work addresses teacher mathematical knowledge of these topics and how it develops, connects this knowledge with classroom instruction and, subsequently, with student learning.

The research is based primarily on a long-term funded project which gathered hundreds of video tapes from diverse middle grades classrooms and thousands of assessments from students. In addition, studies of mathematics knowledge of teachers in training are included, to provide a spectrum of teacher knowledge from those who intend to teach to those with many years of experience. The methods include case studies of selected teachers and mixed approaches which combine detailed descriptions of classroom teaching with quantitative analyses of instruction and student outcomes.

A distinctive feature of the work is its focus on a few specific mathematics learning goals and the use of research-based criteria to describe effective teaching. The focus and in-depth research on equivalent fractions addresses a perennial problem in middle grade student understanding. On the other hand, work on early use of variables in the middle grades represents an emerging issue as mathematics educators discuss the appropriate content and direction for algebra in these grades.

Another important feature is the focus on the use of representations and questions in teaching mathematics. These ideas are at the center of current discussion and research in mathematics education, since they are the primary tools for building student knowledge from concrete to more abstract understandings. Finally, the focus on teacher knowledge in our research is a key idea current mathematics education research. How this knowledge develops and what knowledge is essential for effective teaching, are central questions that we address throughout the book.

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PART I. THE DEVELOPMENT OF MATHEMATICS
KNOWLEDGE FOR TEACHING MIDDLE GRADES
Mathematics knowledge for teaching (MKT) is essential to quality instruction. This chapter describes MKT for key content areas in middle grades mathematics: equivalent fractions, simple algebraic equations, informal geometry, and statistical reasoning. The chapter builds on previous and current research to provide a broad theoretical framework for the research studies that comprise the remaining chapters. Several distinct but related aspects form the basis of our theoretical framework: (1) teacher preparation and teacher learning, (2) teaching middle school mathematics, (3) teacher MKT, and (4) teachers' mathematics knowledge and students' learning.

American classroom instruction has been dominated by a format and procedure that has remained almost unchanged for the past several decades (National Commission on Mathematics and Science Teaching for the 21st Century, 2000). Teachers spend the majority of the class reviewing the previous night's homework assignment and correcting errors, leaving a brief period of time to present and investigate new material. This presentation is often teacher-dominated, with little opportunity for students to explore ideas together or on their own or for teachers to probe students' understanding. Furthermore, the new material is focused on procedures rather than on underlying concepts or on problems that apply the skills. Because little time is spent on concepts and application, students learn poorly, which requires the same routine of review and remediation to be repeated each day. This phenomenon is especially troublesome for the middle grades, in which critical transitions in mathematics learning should take place. Students should move from arithmetic to more abstract ideas and become able to apply mathematical concepts to new situations.

Although the poor achievement of U.S. students has been a recognized problem for many years, the cause was generally thought to be rooted in the curriculum, insufficient time spent on mathematics, or external factors such as TV watching (McKnight, Crosswhite, & Dossey, 1997). The quality of mathematics classroom instruction as a factor affecting achievement was not a major concern of policymakers and educators until it was compared to other educational systems (e.g., Silver, 1998). Classroom video studies of 8th-grade mathematics instruction conducted as part of the Third International Mathematics and Science Study (TIMSS) revealed a teaching gap between American mathematics classrooms and classrooms in other countries (Hiebert et al., 2003; Stigler & Hiebert, 1997, 1999, 2000).
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2004). Most American mathematics teachers taught in similar ways but differently than Japanese or German teachers. U.S. mathematics teachers simply stated mathematical concepts and procedures, which tended to decrease the cognitive demands of the mathematical tasks presented in the classroom. In contrast, teachers in Japan (Stigler & Hiebert, 1997, 2004) and Hong Kong (Stigler & Hiebert, 2004) developed mathematical concepts and procedures and maintained high-cognitive demand in the mathematical tasks employed. Stigler and Hiebert (1997) also reported that over 60% of U.S. teachers believe that mathematical skills (e.g., solving specific kinds of problems or using specific formulas) are important, whereas over 70% Japanese teachers believe that mathematical thinking (e.g., exploring and understanding mathematical ideas or inventing new ways to solve problems) is important.

Efforts to improve mathematics instruction in the U.S. have led to calls for reform-based classroom teaching practices (National Council of Teachers of Mathematics, 1991, 2000). Although there is no one precisely-defined model of reform-based teaching practices (Sherin, 2002), researchers have tried to identify some key features that are believed to be part of effective mathematics instruction. In particular, these features have been identified and articulated as principles in the How People Learn report (National Research Council, 2000a) and Project 2061’s research-based criteria (American Association for the Advancement of Science, 2000). The criteria require curriculum materials and teachers to provide a sense of purpose for students, build on students’ ideas, engage students in mathematics through a variety of contexts and experiences, develop and apply mathematical ideas, promote students’ thinking about mathematics, assess student progress, and enhance the mathematics learning environment. These criteria have been used to guide the analysis of middle grades textbooks (AAAS, 2000) and to design the professional component for prospective (Mathematics Connections Project, 2007) and practicing teachers (AAAS, 1997).

Teacher Preparation and Teacher Learning

National agencies and professional organizations have expressed concern in various reports about teachers’ mathematical preparation. Mathematics courses are needed to help teachers develop a strong and deep understanding of the mathematics they will teach. However, if teachers take more mathematics courses, they could easily move away from the curriculum they will teach, resulting in better preparation for graduate school than for teaching in the classroom (Usiskin, 2001). A recent survey of selected teacher-preparation programs in the U.S. showed few changes in program characteristics related to educational reform recommendations (Graham, Li, & Buck, 2000). In fact, a recent review of research literature on the preparation of mathematics teachers indicated that we do not know much about the effects of prospective-teacher preparation in content and methods on their classroom instruction (Wilson, Floden, & Ferrini-Mundy, 2001). There is general agreement that “to deliver the kind of mathematics content in ways that respects middle grades students as learners demands a well prepared and motivated
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teacher. Few existing teacher-preparation programs meet this need, and certification requirements do not support adequate content and pedagogical preparation” (NRC, 2000b, p. 15). The improvement of teacher preparation must be grounded in research that provides a theoretical understanding of what prospective mathematics teachers for middle grades need to learn, how they learn it, and how their learning can be assessed. This theoretical understanding should be supported with a chain of empirical evidence that depicts prospective mathematics teachers’ learning progress. It also should be based on a real program model, and in turn, it should inform curriculum design and assessment of teachers’ learning of mathematical knowledge needed for teaching. At the same time, researchers also must acknowledge the complexity of this issue, including the diversity of settings and participants and the transition of teachers’ learning in a teacher-preparation program to teaching in real classrooms.

Existing preparation programs for middle grades teachers differ substantially depending on whether they are for generalist teachers in elementary and middle schools or for specialists in middle and secondary schools (e.g., National Center for Educational Statistics, 1995; Nelson, Weiss, & Conaway, 1992). This is partly because of the organization of schools, universities, and state accreditation and certification programs. Moreover, K-8 teachers face a need to integrate mathematics with other content areas that their secondary school colleagues do not face to the same degree. However, there is an increasing emphasis on improving mathematics training for middle grades teachers (e.g., Conference Board of the Mathematical Sciences, 2001; NCTM, 2000). For example, the CBMS, 2001 made the following recommendation:

Recommendation 11. Mathematics in middle grades (grades 5–8) should be taught by mathematics specialists. This recommendation mirrors similar recommendations by a number of other groups seeking to improve U. S. school mathematics instruction. Middle grades mathematics teachers must know the high school mathematics curriculum well and understand the foundation that is being laid for it in their instruction. As concepts like fractions and decimals enter the curriculum, teaching mathematics well requires subject matter expertise that non-specialists cannot be expected to master. Having mathematical specialists, beginning in middle grades, both reduces the educational burden for those teaching mathematics in these grades and provides opportunities for prospective teachers of these grades who like mathematics to specialize in it. (p. 11)

Recognizing that few institutions currently offer programs that target middle school teachers of mathematics, the CBMS report suggests the uniqueness of such a program by asserting that “teaching middle grades mathematics requires preparation different from, not simply less than, preparation for teaching high school mathematics, and certainly reflecting more depth than that needed by teachers of earlier grades” (p. 25). The recommended coursework for teachers of middle school mathematics includes at least 21 semester hours of mathematics. Four broad content areas (i.e., “number and operations,” “algebra and functions,”
“geometry and measurement,” and “data analysis, statistics, and probability”) are recommended. These content areas are also recommended for K-4 teachers but content of the middle school curriculum is extended (e.g., number and operations moves from a focus on whole numbers to a focus on integers and rational numbers) with an emphasis on “making sense, reasoning, and explaining.”

Fully-prepared teachers are more effective in the classroom, and their students demonstrate larger achievement gains than students of teachers who are not fully prepared. This preparation includes an integration of field experiences and coursework to develop prospective-teacher understanding of what the education profession entails. While program formats differ greatly, important aspects of successful teacher-preparation programs include subject matter preparation, instruction in pedagogical knowledge, and the clinical experience (Wilson, Floden, & Ferrini-Mundy, 2001). There is limited research available that informs us about the levels of competency prospective teachers can achieve through their preparation program or especially about how the various components of teacher preparation yield teachers with sufficient mathematics knowledge for teaching to be effective in advancing student performance.

Research on Teaching Middle School Mathematics

Teaching middle grades mathematics effectively involves some important strategies that may not be the same as those employed by elementary or high school teachers. At the middle-grade level, it is important that students experience active forms of learning, such as constructing relationships on their own, extending and applying knowledge to new situations and contexts, reflecting on their experiences with mathematics, expressing what they are thinking and what they know, and gaining ownership of their knowledge rather than simply repeating what they have been told (Carpenter & Lehrer, 1999). The particular classroom activities and teaching strategies that support and develop these forms of learning include challenging and engaging mathematical tasks, appropriate representational tools that bridge concrete experiences to abstract symbols and procedures, and routing strategies that engage and require students to restructure and apply their knowledge. Classrooms that promote this type of active and engaged mathematics learning for all students involve a complex set of social interactions. In this type of classroom teachers and students interact with rich mathematical content in a variety of situations and contexts (McClain & Cobb, 2001). This social and cognitive engagement is essential for promoting equity, making mathematics learning available to all students. Because students may be affected in different ways by this type of instruction, teachers should have clear knowledge of cultural factors (Secada & Berman, 1999).

Some specific instructional and learning factors that can produce cognitive change and understanding of mathematics in middle grades students have been found to characterize high-quality instruction. Research on these factors has provided convincing evidence of their importance in mathematics teaching and learning (AAAS, 2000). In particular, two factors appear to differentiate more
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traditional instruction from teaching that can build understanding. These strategies are: (1) building on students’ prior ideas about mathematics and (2) promoting student thinking and reasoning about mathematical concepts (Kulm, Capraro, & Capraro, 2007).

Building on Student Previous Ideas About Mathematics

Developing conceptual understanding in middle grades students requires paying attention to the ideas (both correct and incorrect) that students have learned in previous grades or on their own (Ball, 1993). The importance of finding and using students’ previous knowledge has been recognized for decades. Ausubel (1968) noted that “the most important single factor influencing learning is what the learner already knows” (p. 16). There are numerous implications of this simple notion. Difficulties that beginning or naïve problem-solvers have, in contrast with more experienced or expert problem-solvers, are often characterized by ineffective use of the mathematical knowledge they have available or are able to access flexibly (Schoenfeld, 1992). If students have underdeveloped connections or representations of concepts or procedures that do not help them to generalize new situations or contexts, their thinking can result in misconceptions (Fischbein, Deri, Nello, & Marino, 1985; Bell, Greer, Grimison, & Mangan, 1989). For example, students’ ideas about number operations need to be revised when they move from whole numbers to fractions and percents (Graeber & Campbell, 1993). Students may think that multiplication always results in a product that is larger than either of the two original numbers. Of course, this generalization (some may call it a misconception) can lead to trouble when working with numbers less than one. Hart (1988) and Matz (1980) found many other examples of knowledge from arithmetic that can lead to misconceptions when students begin work with more advanced topics.

Teachers can improve their instruction if they understand how students learn and think (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Cobb et al., 1991). In order to identify and address prior student knowledge, teachers should have students discuss how they perceive the difference between two representations or solutions to a problem. This discussion can help both the teacher and the students to develop insights into their understanding (Cobb, 1988). Asking students to generalize or extend procedures to new contexts or situations can reveal misconceptions or lack of understanding (Hiebert & Wearne, 1986). These strategies apply to many different mathematical concepts and procedures.

Middle grades teaching should also provide opportunities for students to make connections between and among mathematical concepts and procedures. Resnick (1987) concluded that without explicit instruction and guidance in making these connections, students will not learn to build knowledge. Unless teachers attend to students’ prior knowledge and are alert to gaps or misconceptions, the sequence of activities might not be effective for all students (Mack, 1990). Even worse, further misconceptions may develop, or for many students who do not develop more abstract ideas, performance may be negatively affected. This lower performance
Promoting Student Thinking About Mathematics

Effective classroom discourse can be a powerful strategy, focusing students' attention on thinking about and constructing mathematical relationships (Greeno, 1988; Resnick & Omanson, 1987). If teachers expect students to engage actively in discussing and thinking about ideas, students will have improved understanding of the concepts and nature of mathematics (Lampert, 1989). Work in pairs and small groups can be an effective tool for promoting student communication, which is necessary for reflection and understanding. Both Slavin (1989) and Webb (1989) found that frequent work in small groups can enhance achievement if the work is focused carefully on learning mathematical ideas. However, effective use of cooperative learning in mathematics requires an effective teacher. The most critical component of cooperative learning is the quality of student discussion, which is mainly affected by teacher intervention. Teacher interventions should focus on developing higher-cognitive levels in order to improve students' thinking. Ding, Li, Piccolo, & Kulm (2007) found that teachers varied significantly in the effectiveness of their interventions. Some teachers simply observed groups and kept order, while others intervened by asking probing questions that helped students develop and revise their ideas.

Guidance of student interpretation and reasoning through classroom questions can help students construct and formalize their ideas so they are more accessible. Students need the opportunity for self-discovery through activities that are unstructured enough to allow them to derive generalizations and invent their own procedures (Doyle, 1983). Questions in the lesson summary can also help students reflect on mathematical concepts and help them establish connections between mathematical topics (Madsen-Nason, 1988). The most effective types of questions to promote student thinking are those that guide them along the way and that probe for deeper understanding and reflection (Sahin, 2007).

Mathematics Knowledge for Teaching

The study of teacher knowledge, beliefs, and practices is crucial to understanding and improving pedagogy and classroom experiences (Hashweh, 1996; Nespor, 1987). Research has illuminated the knowledge needed to teach specific mathematical ideas and skills for understanding (e.g., Putman & Borko, 2000; Fennema & Franke, 1992; Hiebert & Carpenter, 1992; Ma, 1999). Mathematics teachers need specialized knowledge that “includes an integrated knowledge of mathematics, knowledge of the development of students’ mathematical understanding, and a repertoire of pedagogical practices that takes into account the mathematics being taught and the students learning it” (NRC, 2001). Likewise, the NCTM has emphasized that teachers need different kinds of knowledge, such as
knowledge of specific content, curricular goals, the challenges students face in learning these ideas, assessment, and effective teaching strategies (NCTM, 2000).

Teachers also need content that is much more coherent than the traditional content presented in widely-used textbooks. For example, Fennema and Franke (1992) consider all aspects of teacher knowledge and beliefs as a system in which teachers transform, interact, and develop new knowledge. Their model challenges researchers to explain “the relationship between the components of knowledge as new knowledge develops in teaching” and “the parameters of knowledge being transformed through teacher implementation” (p. 163). In general, teacher knowledge includes an integrated knowledge of mathematics, students, and instructional practices (NRC, 2001; Shulman, 1987).

Among various components of teacher knowledge, content knowledge is an important aspect of effective teaching, along with the ability to present content in a way that fosters understanding (Popham, 1999). The RAND Mathematics Science Panel (2003) report found a compelling relationship between what teachers can do with their students and their own level of mathematics competence. The obstacle is that “either teachers do not have enough content knowledge, or what they do know is not the ‘right’ content knowledge” (Sherin, 2002, p. 123). This knowledge has recently been denoted by the term mathematics knowledge for teaching. One definition of this term is the following:

Mathematics knowledge for teaching is the mathematical knowledge used to carry out the work of teaching mathematics. Examples of this “work of teaching” include explaining terms and concepts to students, interpreting students’ statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and providing students with examples of concept, algorithms, and proofs. (Hill, Rowan, & Ball, 2005, p. 373)

Making decisions and having the judgment or ability to interpret student thinking means that mathematics teachers not only need to have sufficient content knowledge of mathematics but also knowledge of learners’ cognitions of mathematics. Sowder and Philipp (1999) observed that teachers must themselves have chances to revisit and reconceptualize the mathematics they are teaching and to understand the nature of mathematical and pedagogical knowledge. Content-based pedagogy helps teachers recognize common student conceptions relevant to specific learning goals and to provide tasks and pose questions that guide students’ interpretations of mathematics (NRC, 2001). “There is a positive connection between subject matter preparation (in both content and specific teaching methods) and teacher performance; however, for some subjects, like mathematics, current subject matter preparation (including an academic subject major) may need to be reformed to increase reasoning skills and conceptual knowledge” (Association for Supervision and Curriculum Development, 2003, p. 1). Existing research has deepened our understanding of the kind of knowledge needed for effective mathematics instruction. However, the nature and development of teachers’ mathematics knowledge for teaching still remains largely unknown (Ball, Hill, & Bass, 2005).
Teacher Knowledge and Student Learning

Improving student learning in mathematics requires coordinated reform of curriculum materials, instruction, assessment, and resources. In creating standards documents, both NCTM (2000) and AAAS (1993) drew on a large body of research findings on how students learn particular ideas, when they are best able to learn them, and the difficulties they are likely to have (e.g., Carpenter, Fennema, Peterson, Chiang & Loe, 1989; Cobb et al., 1991; Hiebert & Carpenter, 1992; Owens & Wagner, 1993). Among the most significant findings are those that deal with how students connect new information and concepts to those they already have (NRC, 2000a). Current research has much to say about common student preconceptions and how students restructure their thinking to incorporate new ideas and skills, how they construct new knowledge, and how they transfer knowledge to new situations and applications. These findings have also shaped Project 2061’s research-based criteria for evaluating instructional strategies in the classroom and in curriculum materials (AAAS, 2000).

Sound, high-quality curriculum materials that support quality teaching are an important component. It has been argued that curriculum materials could be agents of instructional change and sources of teacher knowledge (Ball & Cohen, 1996). Project 2061’s evaluation of middle grades mathematics textbooks indicated that the materials funded by the National Science Foundation have high potential for improving student learning (AAAS, 2000). Given the potential importance of curriculum materials (Ben-Peretz, 1990; Freeman & Porter, 1989; Stodolsky, 1989), it is fundamental to understand how they can support teachers in ways that allow the teachers to actually understand and implement effective teaching practices. However, without initial preparation and professional development explicitly aimed at both understanding the intent of these curriculum materials and assessing how to utilize them effectively” it is difficult for teachers to exploit the full potential of these materials for increasing student learning. Professional development by itself, however, is unlikely to produce change, in the same way that curriculum materials alone or teachers alone are unlikely to improve student learning. Interactions among teachers, curriculum materials, professional development, and ongoing support for teachers are what can lead to lasting improvements in student learning.

A large body of research has identified elements of effective instruction in mathematics, including eliciting and responding to student conceptions, structuring inquiry within a series of mathematically-rich problem situations, and promoting discussions among all students in which the validity of ideas lies in mathematical argument rather than the social status of the participants (e.g., NRC, 2000a; AAAS, 1989; Grouws, 1992). However, it is not clear that the majority of teachers make use of these elements. There are few empirical studies on how to develop on a large scale teachers’ knowledge and skills needed to enact effective instructional practices. Mathematical content knowledge is a critical and central component of teacher preparation. Although there has been some progress in the research on teacher knowledge, the conclusion reached by Post, Harel, Behr, & Lesh (1991) remains valid. They stated that “... we really do not know very much about what
mathematics intermediate level teachers actually do know and understand” (p. 183). Teachers’ knowledge-base includes mathematical content (knowledge of the topics of mathematics), epistemology (beliefs and understanding of how students learn material), and pedagogy (the ability of to teach the content), and epistemology the ability to teach in accordance with how students learn) (Harel, 1993). To determine if teachers have acquired the necessary content knowledge requires trustworthy measures of that content knowledge (Floden & Philipp, 2003). A fuller picture of the mathematical understanding of teachers and the means to assess this knowledge is particularly needed.

Number and Operations

Many prospective teachers think they understand rational numbers because they can perform algorithms for computations with numbers and can teach some procedures. Having rational number sense means more than just manipulating numbers. It involves attaching meaning to the symbols used for the rational numbers, moving easily between representations, understanding and comparing relative sizes of rational numbers, making mental calculations, and recognizing reasonable solutions (CBMS, 2001). Middle school teachers typically are only superficially familiar with the content of the multiplicative conceptual field – rational number, multiplicative operator, quantitative reasoning, multiplicative reasoning, and proportional reasoning (Sowder, Philipp, Armstrong, & Schappelle, 1998). Using a written content-analysis instrument, ongoing assessment during seminars, classroom observations, and interviews with the teachers, Sowder et al. (1998) investigated teacher knowledge of the multiplicative conceptual field. They found the teachers’ thinking to be procedurally-oriented, focusing on procedures with little or no focus on understanding the process of working with fractions. In a study of prospective teachers, Ball (1990a, b) found that understanding seemed to be composed of only remembering rules for specific cases.

Although children’s difficulty with interpreting remainders when dividing has been documented (Silver, Shapiro, & Deutsch, 1993), teachers also have difficulty with this concept (Sowder et al., 1998). Prospective teachers’ knowledge of division appears to be based on memorization of rules rather than on conceptual understanding (Ball, 1990a). Wearne & Kouba (2000), in their analysis of results of the National Assessment of Education Progress for rational numbers, found that students at all grade levels had difficulty justifying their responses and explaining how they arrived at these responses. Proportional reasoning, called the “cornerstone of all that is to follow” (Lesh, Post, & Behr, 1988, p. 94), is developed in stages and depends on instruction (Lamon, 1995). Fractions and proportions are central to the mathematics of the middle school program and are the foundation for much of the mathematics beyond this level (Lesh, Post, & Behr, 1988). Essential for understanding the symbolic operations for fractions and decimals is an understanding of the symbolism for representing fractions and decimals (Wearne & Kouba, 2000).
Algebra

Algebra concepts form an important part of the middle school curriculum. Ball (1990b) reported that most prospective teachers, when given an equation to solve, focused on the mechanics of manipulating the equation without focusing on the meaning of the equation. Middle school teachers often need experiences working with functions because the concepts of variable and function are poorly understood even by some calculus students (Harel & Dubinsky, 1992). Linked inextricably with algebra is the concept of function. Meel (1999) studied the responses of 29 prospective elementary teachers to questions focused on understanding the function concept, concluding that many of the prospective teachers held historical definitions tied to formulaic rules, and this negatively affected their ability to solve certain types of function problems. In a study of preservice secondary teachers, Bryan (1999) noted that the group of nine participants, in response to direct interview questions, either offered no explanation or offered a flawed explanation about 35% of the time, compared with a 22% rate of offering a sound explanation. Van Dooren, Verschaffel, & Onghena (2002) determined that future secondary school teachers preferred the use of algebra, both in their own solutions and in their evaluations of student work, even when an arithmetical solution seemed more evident. Some future primary school teachers preferred to apply exclusively arithmetical methods, leading to numerous failures on word problems.

Geometry

The van Hiele model of geometric thought involves five levels of understanding: the visual, descriptive, relational, deductive, and rigorous. The first research related to preservice teachers and the van Hiele model, conducted by Mayberry (1983), found inconsistent understanding throughout the levels. Henderson (1988) examined the impact of this same model of geometric thinking and instructional behaviors to preservice secondary teachers. Battista, Wheatley, and Talsma (1982) studied the importance of cognitive development and spatial visualization for achievement in preservice elementary teachers enrolled in four geometry courses, finding that spatial visualization scores on a standard test improved with spatial visualization activities, leading investigators to believe that activities like paper folding, tracing, symmetry, and so forth might be used in to improve students’ spatial ability. Battista (1990) studied spatial visualization of high school students and compared results with their teachers’ encouragement to improve spatial ability, reaching the cautious conclusion that teacher effect and instructional emphasis might play a role in the development of spatial visualization.

Statistics and Probability

“Of all the mathematical topics now appearing in middle grades curricula, teachers are least prepared to teach statistics and probability. Many prospective teachers have not encountered the fundamental ideas of modern statistics in their own K-12
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mathematics courses, and in fact need convincing that they need to learn this mathematics to be prepared to teach in the middle grades” (CBMS, 2001, p. 114). Other than the work of Watson (2000) and Watson and Mortiz (2002), very little recent research has focused on the probabilistic and statistical conceptions of preservice teachers. Tversky and Kahneman (1974, 1983) laid the groundwork for the study of misconceptions of chance. Many studies have followed their lead, but few have focused on preservice teachers. Similarly, research on the use of measures of center and spread is abundant, but a focus on preservice teachers’ knowledge of these topics is lacking. Shaughnessy (1992) pointed out that “we have to first deal with teachers’ misconceptions before we can expect them to be competent at helping their students to overcome misconceptions” (p. 484). In order to deal with teacher misconceptions, we must first understand the nature of those misconceptions. In addition to the problem of misconceptions, O’Connell (1999) reported that “many college students seem to grasp only a partial understanding of fundamental concepts and procedures in probability” (p. 3). In a recent study, Carter (2005) found that most preservice teachers were able to answer multiple choice and short answer questions on likelihood, data summary, and probability, but very few were able to provide the explanations or interpretations they would need to give to middle grades students.

SUMMARY

As the findings summarized above show, researchers have produced some knowledge about how different elements of the educational system—such as curriculum, instruction, professional development, and ongoing support for teachers—should function to improve student learning of mathematics, but there is little research on how all these elements should work together (Cohen & Ball, 1999; Spillane, Halverson, & Diamond, 2001). By examining the connections among the structure of instructional materials, teacher knowledge, classroom activities, professional development, ongoing support, and student learning, our research seeks to shed light and provide valid statistical evidence on how these elements work together to improve student learning in mathematics. We have created instruments to collect valid teacher and student learning data that can provide information on the conditions under which students learn mathematics well. In particular, we are examining the classroom conditions that enable students to achieve the ambitious learning goals set forth in the new generation of reform curricula. The research we have done takes advantage of the variety of development and implementation efforts that currently exist in mathematics education and addresses key questions asked by educators and the public: Can the reform curricula and teaching really improve students’ learning mathematics with understanding in the middle grades? What teacher preparation and knowledge is required for such learning to occur?
NOTES


REFERENCES


A THEORETICAL FRAMEWORK FOR MATHEMATICS KNOWLEDGE


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A THEORETICAL FRAMEWORK FOR MATHEMATICS KNOWLEDGE


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2. MATHEMATICS KNOWLEDGE FOR TEACHING: 
THE CASE OF PRESERVICE TEACHERS

INTRODUCTION

Reforms seem to be happening all over the U.S. Ongoing education-reform efforts have escalated in response to new and ever-changing accountability systems. Mathematics education continues its “math wars” and is now struggling to put and keep teachers in the classroom. Another reform effort is new teacher education. New teacher education grows out of changing notions of accountability and more specifically, the recent educational reform movements in the U.S. (Cochran-Smith, 2005). The quality of classroom instruction was not been a major concern of policy makers and educators until it was compared to other educational systems (e.g., Silver, 1998). Now the ever-static American classroom (National Commission on Mathematics and Science Teaching for the 21st Century, 2000) is gradually seeing change thanks to numerous professional development programs, nationally-funded research projects in cooperation with major universities, and reform initiatives in teacher preparation programs.

For almost all of the last century, teacher preparation has been located within higher education institutions. This is not the case anymore. Almost every state in the U.S. has some sort of alternative certification program—some are attached to universities while others bypass them altogether (Cochran-Smith, 2005). This has caused colleges and universities to look hard at the way their teachers are prepared. Some programs have been disbanded altogether, whereas others have worked hard to adapt and compete with alternative new teacher education programs. Mathematics education programs have been thrust into the spotlight, especially at the middle-grade levels, because of recent comparison studies revealing poor student performance and static teacher instruction in middle-grade classrooms (Hiebert et al., 2003; Stigler & Hiebert, 1997, 1999, 2004). However, concern about poor student performance or methods of instruction has been overshadowed recently by concern for the mathematical and pedagogical content knowledge of teachers (Hill, Rowan, & Ball, 2005).

In a 2002 study, it was revealed that 68.5% percent of middle-grade teachers have no major certification in mathematics, and 21.9% do not have a minor in mathematics (Seastrom, Gruber, Henke, McGrath, & Cohen, 2002). National agencies and professional organizations in various reports have expressed concern about teachers’ mathematics preparation.
There is evidence of a vicious cycle in which too many prospective teachers enter college with insufficient understanding of school mathematics, have little college instruction focused on the mathematics they will teach and then enter their classrooms inadequately prepared to teach mathematics to the following generation of students. (Conference Board of the Mathematical Sciences 2001, p. 5)

The RAND Mathematics Science Panel (2003) report found a compelling relationship between what teachers could do with their students and their own level of mathematics competence. The obstacle to effective instruction is that “either teachers do not have enough content knowledge, or what they do know is not the ‘right’ content knowledge” (Sherin, 2002, p. 123). The National Council of Teachers of Mathematics (2000) emphasized that teachers need different kinds of knowledge, such as knowledge of specific content, curricular goals, the challenge students face in learning these ideas, assessment, and of effective teaching strategies. In addition, there is a positive connection between subject-matter preparation (in both content and specific teaching methods) and teacher performance; however, for some subjects like mathematics, current subject matter preparation (including an academic subject major) may need to be reformed to increase reasoning skills and conceptual knowledge. (Association for Supervision and Curriculum Development, 2003, p. 1)

Mathematics Knowledge for Teaching

Shulman (1986) introduced the notion of “pedagogical content knowledge,” in which there is a perceived complementary relationship between pedagogical knowledge and content knowledge of a subject area. Hill, Shilling, and Ball (2004) said such specific measures were not yet in place in mathematics education. What they found through their multiple-choice assessment was teachers’ “mathematics knowledge for teaching” (the specific, pedagogical content knowledge of mathematics teachers). Elementary grade teachers’ knowledge was partly domain-specific rather than relating to their teaching or mathematical ability. In their 2005 article, Hill, Rowan, and Ball formally defined mathematics knowledge for teaching:

By “mathematical knowledge for teaching,” we mean the mathematical knowledge used to carry out the work of teaching mathematics. Examples of this “work of teaching” include explaining the terms and concepts to students, interpreting students’ statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and effects of teachers’ mathematical knowledge on student achievement providing students with examples of mathematical concepts, algorithms, or proofs. (p. 373)

In addition to formalizing the definition of mathematics knowledge for teaching, their study of 1st- and 3rd-grade teachers found that teachers’ mathematical
knowledge for teaching was significantly related to student achievement gains in both grade levels. The implementation of the definition of mathematics knowledge for teaching may very well be the beginnings of the reform ASCD (2003) discussed.

STATEMENT OF THE PROBLEM

In order to improve middle-grade mathematics teacher-preparation programs, it is essential to know the nature of mathematics knowledge for teaching in the four main content strands: algebra, probability and statistics, geometry, and number and operations. Not only must teachers understand the material they are teaching, they must also be able to communicate the curriculum to students as well. Therefore, this study utilized a standards- and literature-based assessment composed of randomly selected problems from a database of questions concerning the aforementioned content strands.

The overall purpose of this mixed-methods study was to develop an online assessment that effectively evaluated and allowed preservice middle-grade mathematics teachers to demonstrate their mathematics knowledge for teaching in the four content strands of algebra, geometry, probability and statistics, and number and operation. In addition, this study examined differences in mathematics knowledge for teaching in three different cohorts of preservice middle-grade mathematics teachers. The ultimate goals of this study were to provide an effective online assessment instrument, to provide data on preservice middle-grade teachers’ mathematics knowledge for teaching, and to provide recommendations that may be used to help shape reform initiatives in teacher-education programs throughout the U.S.

METHODOLOGY

This study employed a concurrent triangulation mixed-methods strategy in order to address the aforementioned purposes. The concurrent triangulation strategy uses separate quantitative and qualitative methods in order to strengthen one method in areas where the other method is inherently weak (Creswell, 2003). The author felt it necessary to include a qualitative measure in addition to a quantitative measure in this study to gain a deeper understanding of preservice middle-grade teachers’ mathematics knowledge for teaching. Although connecting test scores of mathematics knowledge for teaching to various course sequencing, cohorts, classes, and other variables could easily be done with quantitative measures, the author felt the study could be enhanced with qualitative data such as open-ended responses to assessment items. Figure 1 gives a more detailed picture of the concurrent triangulation strategy (Creswell, 2003) used in this study. The data were analyzed qualitatively and quantitatively. The open-ended responses were analyzed qualitatively using constant comparative analysis (Denzin & Lincoln, 2000). The data were quantitatively analyzed using univariate and multivariate statistics as well as nonlinear regression.
Figure 1. Concurrent triangulation mixed methods design (adapted from Creswell, 2003; Tashakkori & Teddlie, 1998).

Population

This study occurred at a large public university. The population for the study was preservice teachers pursuing Mathematics/Science Specialist degrees and who were enrolled in one or more of the sequence of mathematics courses within the middle-grade specialist degree program. Of the 122 participants in the sample, 109 (89.3%) were female. Of the participants, 2 (1.6%) were freshmen, 17 (13.9%) were sophomores, 45 (36.9%) were juniors, and 58 (47.5%) were seniors. Of the participants, 3 (2.5%) were Asian or Pacific Islander, 2 (1.6%) were African American, 11 (9.0%) were Hispanic/Latino, 106 (86.9%) were White (non-Hispanic), and none were American Indian or Alaskan Native. None chose not to respond.

Instrumentation

The primary data-collection tool was the Mathematics Knowledge for Teaching Middle Grades Online Assessment given to the targeted population. The assessment instrument was developed by the author and was comprised of four subtests, each targeting one of the content strands: algebra, number and operations, geometry, and probability and statistics. Each of the four subtests consisted of seven questions with three parts apiece. The four subtests were a compilation of released items from the 2005 7th- and 8th-grade New York State Assessment. In developing the assessment, multiple choice answers were removed and replaced.
with three open-ended questions. The first question simply provided a space for the participant to type the answer to the question. All of the items, with the exception of one, had numerical answers. The second part of the item asked for the participant’s explanation of his or her answer. These first two items were aimed at gathering the participant’s content knowledge of the strand addressed. The third part of the item addressed the participant’s pedagogical content knowledge of the content strand addressed. The participant was asked to respond to this question: “How would you explain, model, and/or demonstrate this item to someone who did not understand?” In addition to content knowledge and pedagogical content knowledge, the communication and vocabulary were also of interest to the author.

A content validation study was not done since these items were published, state assessment items created by a major publishing company. The test content was considered valid since it was designed to test the specific New York State standards for 6th-8th grade. The Feldt-Raju index reported for the 2005 Eighth Grade New York State Assessment was 0.94, which is a number comparable to that of the 2004 Eighth Grade State Assessment (CTB/McGraw-Hill, 2005). The standards for grades 6th through 8th in the State of New York align closely with the Texas Essential Knowledge and Skills (TEKS). In addition, both sets of standards align with NCTM’s Principles and Standards for School Mathematics (2000) for their respective grade levels. The Spearman-Brown prediction formula was used to calculate the reliability of each assessment. The coefficient alpha for the number and operations assessment was .924, for the algebra assessment was .935, for the geometry assessment was .973, and for the probability and statistics assessment was .907.

The assessment instrument was administered using the Form Management System (Strader, 2006), which allowed for online data collection. From the time the instruments were posted until they were removed, participants and instructors had 24-hour access. Therefore, the assessment could be completed at the convenience of the participants. Students had access to the test for three weeks. Form Management System (Strader, 2006) randomly assigned one of six versions of the assessment to each student upon access to the website (after the consent form was accepted). Each student took an assessment containing eight demographic questions and two content strands. Each content strand (algebra, number and operations, geometry, and probability and statistics) contained seven questions. Therefore, each student took an assessment with a total of 22 items. Students were allowed to use calculators but no other aids on the assessment.

Scoring

All of the data collected on the assessment, with the exception of the demographic information, were in open-ended question format. Therefore, in order to conduct the quantitative analysis for this study, the responses were scored by hand. The first part, answering the content question, was scored right or wrong, indicated by a 0-wrong or 1-right. The second part, explaining the solution to the problem, was scored on a 7-point holistic rubric. The third part, explaining the item to someone
who did not understand, was scored on a similar 7-point holistic rubric. The weights of the scores were determined using Ball’s (2006) definition of mathematics knowledge for teaching. The framework (shown below in Figure 2) suggests common content knowledge is just a small part of subject matter knowledge, therefore, part one of the assessment was scored as 0 or 1. Parts two and three of the assessment appear to be equally weighted in the diagram, therefore, they were scored on a 7-point scale, adapted from a 4-point rubric published by the New York State Testing Program (2005). Intra-rater reliability for grader 1 was 97.7%, Intra-rater reliability for grader 2 was 96.7%. Overall inter-rater reliability was 99.3%.

![Figure 2. The assessment of preservice middle-grade teachers’ mathematics knowledge for teaching (adapted from Ball, 2006).](image)

**RESULTS**

Analysis for the four content strands was broken into four parts: analysis of content knowledge, analysis of content understanding, analysis of pedagogical understanding, and analysis of the mathematics knowledge for teaching. The three parts—content knowledge, content understanding, and pedagogical understanding— together make up mathematics knowledge for teaching the content strand.

Item Analysis was conducted for all four tests’ content knowledge. The average percent correct for number and operations was 56%, for algebra 79%, for geometry 76%, and for probability and statistics 76%. These percentages are considered moderately difficult on an item difficulty scale. Although an item difficulty of moderately difficult is usually deemed good for an assessment, these questions
were taken from a middle-grade standardized assessment. Therefore, the content questions should have been easy for the participants to answer.

**Mathematics Knowledge for Teaching Number and Operations**

*Number and operations content knowledge.* Items on which less than half of the participants answered correctly were investigated for commonalities. The first item on the number and operations assessment addressed developing and applying the laws of exponents for multiplication. There were two common errors found in answers to this item. The first error was after the student correctly multiplied, the solution was left in incorrect scientific notation. The second error occurred when the student multiplied correctly, but the coefficient was a number less than ten. Two items on the number and operations assessment addressed estimating a percent given an application. The most common error on both of these items was simply not estimating. A majority of the students who missed this item reported the exact number instead of the estimate (i.e., $562.50$ instead of $550$ or $600$; or $38\%$ instead of $40\%$).

These responses could indicate a lack of understanding of the properties of scientific notation; or simply carefully checking the answer. The failure to estimate could have been a failure to read the instructions carefully or could indicate a lack of understanding estimation strategies. Several of the students in the content understanding responses indicated they estimated the answer after they performed the actual operation. This indicates that the failure to estimate and to estimate properly could be due to a lack of understanding.

*Number and operations understanding.* Responses to open-ended questions that assessed content and pedagogical understanding of the number and operations were explored using constant comparative analysis (Denzin & Lincoln, 2000). In their explanations of solutions to items on the number and operations assessment, a majority of the participants scored either a two or three according to the rubric for this item. A score of three on the explanation of the solution generally meant students explained their exact algorithmic or mental mathematical procedure. The procedures were mathematically correct and their solution was correct, however, there was no further explanation provided other than their algorithmic procedure. A common example of a score of three:

Multiplied $5$ by $5.9$, and got $29.5$. Added the exponents together to get $10^{15}$. Added one more to the exponents when I moved the decimal point over one space to get $2.95 \times 10^{16}$.

A score of three indicated that the participants did nothing above and beyond what was asked of them. It could be argued that explaining is not simply writing out an algorithmic procedure. However, if the solution was correct and was mathematically sound, the student received a minimum score of three according to the rubric. A score of two on content understanding for number and operations items generally meant students had a mathematically sound procedure; however, their answer did
not match what the problem asked. The following statement is mathematically correct; however, the question asked for an estimate rather than an exact answer.

Subtracted 511 from 705. Divided that number by 511, then multiplied it by one hundred.

Two other items on the number and operations assessment are noteworthy due to common themes noticeable across the explanation of solutions, especially in the misuse of mathematical vocabulary. The first item addressed applying percents to sale price situations. A common error among the incorrect content answers was adding the two percentages given in an original price and then a sale price situation. A common answer for the explanation of the solution was:

Add the total percent discount and then multiply by 37 then subtract that from 37.

A more detailed explanation of the same item:

If you add the percentages off together, you get 35 percent off. So then I broke it down into 30 percent and 5 percent. 30 percent is 10 percent 3 times and 10 percent is 3.70 so I multiplied 3.70 by 3. Five percent is half of 3.70 so I added this all together and subtracted from 37 to get the new price.

This is a common error often found in sale price situations. The students think they can reduce the amount of work they do by simply adding two percents. They do not realize one discount applies to the original price and the other discount applies to the sale price. The next item of concern addressed determining least common multiples of two or more numbers. An overwhelming majority of the students who missed this item mistook the least common multiple for the greatest common factor. Examples of common explanations include (each of these students reported 3 as their final answer):

3 is the smallest number that evenly “goes into” 3, 6, and 27

Knowing what the multiples of all those numbers were because they were small helped me to determine what number was the smallest multiple of all three

It is the lowest value they all have in common

On this same item, misuse of mathematical vocabulary was evident in the explanation of their solution, although the student received credit for a correct answer on the content (part 1). Common explanations included:

Factor then multiply by all the factors.

To have 27 as a factor, the number had to be bigger than 27, and to have 6 as a factor it had to be even. The best way to get a number that fits those criteria is to multiply 27 by 2.

27 times 2 because 27/6 is not a whole number.
The greatest common factor is found when adding and subtracting fractions in number and operations as well as algebra settings. Least common multiple is a term introduced around the 4th or 5th grade and used throughout the rest of the participant’s schooling. A way to show the proper relationship between factors and the least common multiple is to find the prime factorization of all the numbers given and then find the least common multiple based on the number of times a factor occurs in each number. Only two participants on the number and operations assessment explained their solution in this way.

**Number and operations pedagogical understanding.** Many of the same themes and trends found in the explanations of the solutions were found in responses to questions that assessed understanding. If students did not receive a correct answer on the original content problem or received a low score on the explanation of their solution, they were not likely to receive a high score on the pedagogical understanding part. A majority of the participants scored a two, three, or four on the pedagogical understanding part of the number and operation assessment. A score of two for pedagogical understanding of number and operations generally meant the procedure was somewhat correct, but contained minor mathematical errors. A score of three generally indicated students could successfully explain their procedure to a person who did not understand. Scores of two and three on the pedagogical understanding part tended to be very algorithmic in nature, and the general method of instruction could be assumed to be direct because there was no mention of any other method. Examples of scores of two and three on the understanding part of the number and operations assessment follow:

1. I subtracted 511 from 705 to get 194. I then divided 194 by 511 to get .37964 and then multiplied by 100 to get the percent of about 38.

2. I’d explain the definition of exponents and how $4^3$ is short hand for $4 \times 4 \times 4$. And then work the multiplication.

A score of four on the pedagogical understanding part indicated the student could do more than just explain their algorithmic procedure. They could relate their procedure back to different parts of the problem and could often explain why they got a certain number or conducted a certain procedure. It is also at this level that the researcher first saw signs of cultural responsiveness on the student’s part: mention of various instructional techniques, application to real world situations, and use of hands-on learning. Here are examples of a score of four on the pedagogical understanding part of the number and operations assessment:

When we solve percent problems, we set up a proportion in words first. The words are PART over WHOLE equals PERCENT over 100: part/whole = %/100. Look at the first part. The shirt cost 37. Is that the whole cost or part of the cost of the shirt? The whole. And we want to know what 20% is... so we have 7/37=20/100. Divide 100 by something to get 37. Do long division and figure out that something is 7.4. So the 20% sale gives him $7.40 off the price. Subtract that from the cost and now the shirt costs $29.60. Then we set
it up the same way and try to find what 15% of the sale price is: it’s $4.44. Subtract that from the new cost to get the final cost of: $25.16.

You could use the colored rod method. Build three trains of 3, 6, and 27 in length. Keep building each train using trains of the same length until all three trains are equal. The length of the equal trains is the least common multiple.

The three parts of the assessment (content knowledge, content understanding, and pedagogical understanding) make up mathematics knowledge for teaching. Total scores for this assessment ranged from 3 to 59, a range of 56. Total possible points for the entire number and operations assessment was 91 points.

Mathematics Knowledge for Teaching Algebra

Algebra content knowledge. There was only one item on the algebra assessment that less than half of the participants answered correctly. This item addressed translating verbal sentences into algebraic equations. There were two common errors found in student answers to this item. The first error was simply not following the directions of the item. The item asked the student to set up an inequality for the given information. About half of the students who missed this item solved the inequality and reported an answer instead. The second error was the misinterpretation of the information given in the item. The item stated, “The website fee is $30.” The inequality was to be used to determine a minimum profit. The other half of the students who missed this problem mistook the website fee for a profit and added 30 to the inequality instead of subtracting the fee. This mistake could be due to participant lack of problem solving skills. Carefully reading items and carefully extracting all given information properly are essential to effective teaching in the middle grades.

Two other items concerning incorrect answers on the content knowledge part of the algebra online assessment are noteworthy. Item five addressed translating verbal sentences into algebraic equations. Of the three items on the assessment addressing this idea, this item was the only one not presented in a real world context. The one major error common to a majority who answered this problem incorrectly was the incorrect placement of the “eight less than” in the equation. Instead of subtracting eight from “twice a number” (i.e., \(2x - 8\)), students subtracted the “twice a number” from the eight (i.e., \(8 - 2x\)). Item six addressed factoring a trinomial in the form of \(ax^2 + bx + c\). Of the students who missed this item, with the exception of those who did not complete the item, all simply mixed up their signs when reporting their factored answer (i.e., \((y + 3)(y - 6)\) instead of \((y + 6)(y - 3)\)). Using the distributive property, participants could have checked their answer. Checking the answer is an essential part of the problem solving process and is often the part omitted. Several of the participants stated they multiplied their answers back out to check themselves. Interestingly, a majority of the participants who switched their signs and used the distributive property to check their answer got the original problem. The reason for the higher scores in the area of algebra
could be from the concentration on algebra often found in high school and early college courses in mathematics.

In terms of content understanding (part 2) of items on the algebra assessment, a majority of the students scored either a two or three. A score of three on the explanation of the solution generally indicated that students simply explained or algorithmically showed their exact mathematical procedure for the given item. The procedures were mathematically correct and their solution was correct, however, there was no more explanation provided other than their algorithmic procedure. Common examples of a score of three:

Subtract the second parenthesis from the first. Look for like terms. Subtract like terms to finish the problem.

A score of two on the explanation of the solution (content understanding) for algebra items generally meant the student had a mathematically sound procedure; however, their answer did not match what the problem was wanting. The following example of an explanation is of a student who got a correct answer for the item but misused mathematical vocabulary, gave a limited explanation, and therefore scored a two:

I multiplied each factor using FOIL.

Five of the seven items on the content understanding part of the algebra assessment are noteworthy. Three of the items addressed translating verbal sentences into algebraic equations. Although a majority of the students got the items correct, they had trouble explaining their solutions. A common theme across all explanations for these items was “just follow the words... they tell you what to do.” Common examples of explanations for these items include:

Well you really just have to be familiar with the terminology of a word problem or if you know how to solve the problem on your own you could do that. But the word problem practically tells you the equations; you just have to be able to put it all together.

All I did in this problem is read the statement and translate what it was saying into numbers and operations.

It is quite simple. All it is is that the number of cars making $4 per car minus the fee must be at least $50 for him to make a profit.

Again, this could tie back to problem-solving processes. Problem solving is an essential component of the middle-grade mathematics curriculum and is a strand in the NCTM (2000) standards. Participants take a course in problem solving (MASC 351) during their sophomore or junior year, which could help with problems and explanations such as these.

The other two items complement each other in that one addressed the multiplication of binomials and the other addressed factoring a trinomial. In the first item, the common explanation was “I just used FOIL to multiply it out and then I added like terms.” A another example of a common solution was:
I came to this answer by foiling or multiplying the two parentheses together. By doing this I get $6x^2-24x-10x+40$. Then combining like terms I get $6x^2-34x+40$, which is my answer.

Another method mentioned, but not as frequently, was the “box method” or “tic-tac-toe method.” Students who mentioned this method in their explanation of their solution tended to score lower because they were unable to clearly communicate this procedure:

I did this problem using a tic tac toe method. You draw what looks like a tic tac toe board and then put the first binomial across the top and second binomial down the side. I will try to demonstrate what this looks like on paper but without the lines: 

\[
\begin{array}{cc}
2x & -8 \\
3x & 6x^2 -24x & -5 & -10x & 40 \\
\end{array}
\]

There would be lines separating the lines and columns. Next you combine like terms. -24x plus -10x can be added together and nothing else can. This gives you our final answer: $6x^2 -34x +40$

A lack of the phrases “distributive property” and “multiplication of each term” among the majority of the explanations was noted. Although a majority of the students answered item six correctly, there was a tendency to misuse mathematical vocabulary and procedures in explanations of solutions. For example, several students explained that they used “FOIL” to factor their trinomial:

The solution is $(y - 3)(y + 6)$ because when you FOIL these two factors together it gives you $y^2+3y-18$. FOIL is just a way of multiplying the two factors together without leaving out a part.

$y$ is squared so it needs to be in both parentheses. Then use FOIL.

Across item six explanations, the most common method of arriving at a solution was to work the FOIL method backwards:

I did the FOIL method backwards in a way. I know $y^2$ is $y$ times $y$ so that will be the first portion of each binomial. Then I thought about the factors of 18 and picked on that when subtracted would equal 3.

I got this by working backwards from my original polynomial using the FOIL method. I knew that I needed each of my binomials to have $y$ as the first term so that they would give me $y^2$ as my first term in the polynomial. I then looked at what two numbers multiplied together would give me 18, that would also give me a -3 when added together. I came up with 6 and 3, and I knew that to get a +3, the 6 had to be the binomial with the + sign and the 3 with the - sign. Thus giving me $(y+6)(y-3)$.

Although the explanations above considered the $y^2$ term, this was not common to most answers. Most of the explanations just concentrated on explaining how the factors of 18 were combined to find the middle term:

I just did this problem in my head by breaking it down into what factors of 18 could be subtracted to give me a difference of 3.
I arrived at this solution by deciding the factors of 18 and then once I did that I knew one would be positive and the other negative because the 18 was negative. All I had to do then was figure which two values subtracted from each other was +3. The solution to this is +6 and -3 which when added equal +3 and when multiplied equal -18 which we are looking for.

FOIL is a process that is supposed to aid students in using the distributive property to multiply two binomials together. However, from these examples, one can see that participants memorized the procedure and may not understand what they are doing. FOIL (or other methods such as the box method) is taught as early as 6th grade and ingrained into students' minds throughout high school and even into college. There are alternative procedures for teaching the multiplication of binomials that the participants are taught during the middle-grade program at the site of this study, including algebra tiles.

Algebra pedagogical understanding. Many of the same themes and trends found in the explanation of the solution emerged. If students did not receive a correct answer on the original content problem or received a low score on the explanation of their solution, they were not likely to receive a high score on the pedagogical understanding part. A majority of the participants scored a two or three on the pedagogical understanding part of the items on the algebra assessment. A score of two on the pedagogical understanding part of the items on the number and operation assessment generally meant the procedure was somewhat correct but contained minor mathematical errors or missing parts. A score of three generally indicated students could successfully explain their own procedure to a person who did not understand. Scores of two and three tended to be very algorithmic and procedure-oriented in nature. The general method of instruction could be assumed to be direct since there was often no mention of any other method. Examples of scores of two on the understanding part of the algebra assessment include:

- They would need to know a little bit of geometry and understand the meaning of transversal and what that means in regards to parallel lines.
- I would show them the FOIL method and how I went about solving this particular problem using it.

Examples of scores of three include:

- After reading “eight less than twice a number is forty-two” first we know that the equation = 42 and we know that twice a number is 2xN or 2N. Eight less of 2N is 2N-8 which equals 42.
- Use the method First—multiply the first set of numbers in the parentheses and carry the values, outside—multiply the outside sets of numbers, inside—multiply the inside set of numbers, last—multiply the last set of numbers. Then add or subtract the 2nd answer and 3rd answer you got for the outside part and inside part. FOIL
Across all the pedagogical understanding responses, there was a greater indication of cultural responsiveness on algebra items than was found on the number and operations items. Exemplar pedagogical understanding explanations include:

I would write $3x(2x-8) + -5(2x-8)$ and draw arrows from the coefficients outside the parentheses to the two characters inside of each.

I would bust out a protractor to show that the angles are indeed equal, and proceed in a similar manner as my explanation of my solutions. If someone did not understand the algebra, I would do a mini-lesson on doing like-operations on both sides of an equal equation in order to solve it.

I would bring in a 1 gallon jug and demonstrate the equation using the information from the problem. We could say the capacity of the jug is 12 gallons, but we would only fill the container partially. Then, calculate how much it would cost to fill the rest of the jug.

It should also be noted that there were a few examples of multiplying binomials where the participants stated they would use Base-10 blocks to model this procedure. Only one participant correctly incorporated the use of a hands-on material in the multiplication of binomials. The participant suggested algebra tiles be used to model this item. Base-10 blocks are not effective because they all fit together. Algebra tiles have an odd length for the $x$-bar and the unit squares do not fit evenly across the bar. This is to help represent $x$ as an unknown. Base-10 blocks cannot do this.

Other themes noteworthy for pedagogical understanding in algebra were the teaching of the concepts of the multiplication of binomials and the factoring of trinomials. Again, FOIL, forwards and backwards, was a common theme in the responses. A major concentration on just finding the correct combination of factors in factoring trinomials was noted. Common examples for the multiplication of binomials include:

Ok, I want to teach you FOIL. Front, Outside, Inside, Last. SO when we see a problem like this one we know what to do. Front: $3x\cdot 2x = 6x^2$ Outside: $3x\cdot -8 = -24x$ Inside: $-5\cdot 2x = -10x$ Last: $-5\cdot -8 = 40$. We have $6x^2 - 24x - 10x + 40$. Now we collect like terms and end with $6x^2 - 34x + 40$.

Explain the FOIL method means first, outer, inner, last. So we must first multiply the first variables in each expression. Then we multiply the outer numbers, then the inner numbers, and then the last two variables in each expression. After we’ve found the solutions we combine our like terms, (usually found with the inner and outer answers) and add all of them together. I would make sure to tell the students to watch out for negative signs!

Common examples for the factoring of binomials include:

I’d tell the kids to automatically put $(y)(y)$, because we know $y\cdot y$ is $y^2$. Then I’d show them that since the second sign is $–$, one sign must be $+$ and
one -. Then I’d tell them to factor 18, find the numbers that will equal 3 when
subtracted, since it’s positive three, you know the larger number goes with
the plus sign and thus we get our answer.

I would have them find factors of 18 and then those factors must somehow
(either by adding or subtracting) equal 3.

Review the FOIL process with models and arrows then help them work it
backwards.

The box/tic-tac-toe method was again men tioned. As with the explanation of the
solution, there was no clear explanation of how the method was to be used:

There are many types of ways to model factoring. I was taught using a box
method, cut into 4 quadrants and is filled in accordingly, with the variables
that the rows and columns have in common on the outside of the box. I
would then make sure they knew to go back and check their work to make
sure they did it correctly!

It was again noted that there was a lack of use of the term “distributive property”
and “multiplication of each term.”

All three of the parts (content knowledge, content understanding, and pedagogical
understanding) make up the component called mathematics knowledge for
teaching algebra in the middle grades. Total scores for this assessment ranged from
4 to 80, a range of 76. The total possible points for the algebra assessment was 91.
The mean was used to calculate the averages; it is important to remember the
students are at various points in their program. Since this study is the first of its
kind, no comparisons can be made. Instead, these numbers are setting the bar for
future studies.

Mathematics Knowledge for Teaching Geometry

Geometry content knowledge. There were no items for which less than half of the
participants answered correctly, therefore the three items with the smallest
percentage of the participants answering correctly were investigated for
commonalities across each item. The first item addressed calculating the missing
angle in a supplementary pair. Participants who answered incorrectly most often
identified their answer as 58 degrees instead of 60 degrees. This is a minor
“plugging in” error. The item requires one to solve an equation for \(x\). However, the
item asked for the measure of an angle, given an expression \(x + 2\). This “plugging
in” error often happens when there is more work to be done after an equation has
been solved.

The second item addressed identifying pairs of complementary angles. In this
item, the majority of those who answered incorrectly reported angle \(Q\) as the
complementary angle to the given angle \(X\). Angle \(Q\) is actually vertical to angle \(X\).

The third item addressed determining angle relationships when given two
parallel lines cut by a transversal. A majority of the participants who answered
incorrectly described the relationship between the two angles as equaling 180
degrees. This is partially correct, but the researcher was looking for the specific term, supplementary. It was noted that a few of the responses to this item reported a complementary relationship instead of a supplementary relationship.

Geometry content understanding. In explanations of solutions, a majority of students scored three according to the rubric, indicating that the students explained the algorithmic or mental mathematical procedure for the given item without additional explanations. The procedures were mathematically correct and the solution was correct. Common examples of a score of three on the geometry explanation part include:

I combined both A and B expressions and set them equal to 180. I solved for the missing variable, plugged that back into the expression for angle A and got the answer.

Vertical angles are basically opposite angles. So since 2 is opposite of 4, they are vertical and since 1 is opposite of 3, they are also vertical.

Overall, the explanations of the solutions for the geometry assessment revealed incorrect mathematical vocabulary. When dealing with parallel lines and a transversal, a majority of the students mixed up terminology for corresponding angles, alternate interior angles, and alternate exterior angles. In addition, several students thought “complementary” and “congruency” were analogous terms:

By looking at the picture you can see that m<R=90 by opposite angles of a line. You can see that m<P = m<S by opposite exterior angles. So then we can see that Q is equal to X. (Item 3— the angles are actually supplementary and the item asked for complementary angles, not congruency)

I just looked for the angle that was directly opposite of the angle X (Item 3)

Corresponding angles are angle 1 and angle 2 (Item 4— the angles are actually supplementary)

Opposite exterior angles are congruent. (Item 4)

Vertical angle terminology revealed that students often associated vertical angles with longitudinal directions instead of intersecting lines and their subsequent angle relationships. Other common terms used instead of vertical angles were opposite and diagonal:

a is diagonal from 51, so I assumed it was also 52, and then subtracted from 180 to get 129 (Item 7)

Vertical means up and down. Horizontal means side-to-side, so the angles that are facing up and down are 2, 4. (Item 5)

Vertical angles would be the ones that are going up and down... like north and south (Item 5)
They are opposite angles of each other so they are equal which implies they are congruent. (Item 6)

It is unclear why there was so much confusion with the vocabulary associated with the geometry items. Vertical is a directional term in everyday life, and the participants may not have seen geometry material since their early years of high school. So the participants may have forgotten much of the terminology and relationships of geometry.

Geometry pedagogical understanding. A majority of the participants scored a three or a four on the pedagogical understanding part of the geometry assessment. A score of three generally indicated that students could successfully explain their own procedure to a person who did not understand. A score of four indicated the student could go beyond the explanation of their own procedure, many times adding terms and definitions associated with the problem or providing a basic definition of why they did what they did. An example of a score of three follows:

If they knew what vertical angles were then I would remind them that vertical angles are congruent and therefore, S&T and R&U are congruent because they are vertical angles.

Below is an example of a score of four:

I would teach supplementary angles, then parallel lines and corresponding angles. I would show the relationship between all these and then explain the solution above like that.

Although directional association with vertical angles was prevalent in the explanation of solutions, there was only one directional association in the understanding part of the problem:

Show a map or some other object with north and south orientation and then demonstrate how the picture is similar.

Of the students who mentioned directional associations in the explanation (content understanding) part, a majority just mentioned describing the definition of vertical angles to someone who did not understand. There was no additional explanation given as to what this definition might be. There was again an evident misconception that congruence and complementary are analogous terms. The following explanations refer to an item that asked students to identify the complementary angle to $\chi$:

Find the angles of each with numbers and show that they are equal

I could demonstrate this very problem to my class and help them understand why the two angles are equal.

Across all pedagogical understanding parts of the items on the geometry assessment there were several instances of exploration activities. Most of these activities involved providing actual measurements and protractors to students so they could measure the angles and then come up with their own idea of the
relationships between angles (i.e., supplementary, complementary, corresponding, etc.). In addition, a majority of the responses contained some kind of definition or discussion of giving and/or explaining definitions necessary for students to understand the problems. Common examples include:

Have the students explore angles and what it means to be supplementary. Have them measure the angles formed on both sides of an intersecting line. This should result in an understanding that no matter what the measure of the angle \( (x) \) is on one side of a intersecting line, the measure of the angle on the other side of the line is \( 180-x \). Now have the students use this knowledge and armed with the definition of supplementary to tackle this problem. If the two angles are supplementary, what does this mean? (Their measure adds up to 180) How would you write this using the above angles? \( m<PQR + m<STU = 180 \). If the measure of \( <PQR \) is \( 2x \), what is the measure of \( <STU \)? \( 2x + <STU = 180 \) which means \( 180 - 2x = <STU \)

After students have investigated real world examples of angles, then pictorial representations of angles have students measure a lot of angles and come up with the relationships between them. They should come to the realization that the measure of the angle on one side of an intersecting line is the same as that of the opposite angle (the vertical angle). Define the term “congruent angles.” Stress that congruence “- angles having equal measure” relates the measurement of the angles, not the size of the representation. Ask the students which angles have the same measure. What does having the same measure mean? So...which angles are congruent?

The three parts (content knowledge, content understanding, and pedagogical understanding) comprise mathematics knowledge for teaching geometry in the middle grades. Total scores for this assessment ranged from 0 to 81, with a range of 80. There were 91 total possible points for the geometry assessment.

**Mathematics Knowledge for Teaching Probability and Statistics**

**Probability and statistics content knowledge.** Only one item was answered incorrectly by more than half of the participants. This item addressed determining the probability of dependent events. There were three common errors. The majority of the participants who did not answer this question correctly simply forgot to reduce the fraction for their final answer. It could be argued that the result did not need to be reduced as it was a probability question. The other two types of errors were adding the two probabilities and the treatment of the item as “with replacement,” even though the problem directly stated “without replacing the first cookie.” Errors such as these reveal a misunderstanding in the concepts of independent and dependent events.

**Probability and statistics content understanding.** A majority of the participants scored a three according to the rubric for this part of the assessment. A score of three meant students explained their exact algorithmic or mental mathematic
procedure for the given item. The procedures were mathematically correct and their solution was correct, however, there was no further explanation provided other than their algorithmic procedure. A common example of a score of three follows:

Since there are 21 possible options for the first letter, 5 for the second, and 21 for the third, you just do 21*5*21 and that gives you 2,205 possible tag codes.

There were a few items with minor errors in explanations of noteworthiness. The first is an item that addressed calculating the range for a given set of data. Of the students who answered incorrectly, a majority of them indicated the range of data as an actual range (i.e., 73 - 97) instead of calculating the range by subtracting the highest and lowest scores. This may have resulted from real life experiences where the range is often reported, “the test scores ranged from 73 to 97.” Another common error was the ordering of the numbers. The numbers were presented in non-numerical order and instead of reordering them, students took the last score minus the first score. This result may reveal a misconception that the range is the difference between the first and last scores.

One of the items addressed reading and interpreting data represented graphically through a pictograph. This item is the only item that received 100% correct responses out of all four assessments. However, after a comparison of the explanations, it was revealed that although the participants correctly chose yellow, their explanations were not correct. The pictograph displayed three times greater sales for yellow than brown. However, participants misinterpreted three times as three more. Green has three less or two times fewer sales than yellow. This could be from a misunderstanding of the common vocabulary associated with multiplication and addition. Common examples of this misinterpretation include:

Yellow has three more sunglasses in the pictograph than green, so that is how I got my answer.

You would look at the graph and start with the smallest number of sunglasses, which is brown. The pictography has only two sunglasses (20,000) for that. The next one up, green, has three sunglasses (30,000). You can see that that is not the answer, because you are looking for something that is three times greater, which means it needs to have three more sunglasses in the pictograph. If you compare all of them, you find out that yellow has three more (30,000) than green. So, yellow is the answer.

Yellow had sales three times greater than green because there are three more sunglasses pictures for the yellow than there are for the green.

Another item addressed predicting the outcome of an experiment. Of the students who incorrectly answered this problem, a misinterpretation of the given graphic representation was the most common error. The representation contained ten numbered cards with the color written on them. The black cards were darkened a deep gray while there was little room for distinction between the white and gray
cards. Because they looked at the printed color of the card instead of reading the color, participants reported probabilities of 7/10 instead of 4/10 in order to predict the outcome of the experiment:

Because every single time he has 10 cards to pick from and every single time there is a 70% chance (7 out of 10) that he will pick a white card. Conducted 100 times, the theoretical number of times that he should pick a white card is 70. That said, it will probably not be the actual number.

There is a probability that Derek will pick up a white card 7 out of 10 times. You multiply this 100 times. You don’t have to change the number because he replaces the card each time.

This again falls back to the problem solving process; failing to read the problem carefully and not checking the answer to assure it makes sense.

Probability and statistics pedagogical understanding. A majority of the participants scored either a three or a four. A score of three generally indicated students could successfully explain their own procedure to a person who did not understand. Common examples of a score of three on various items within the probability and statistics assessment include:

I divided 240 by 3 since 1 through 5 contains 3 odd numbers.

See if they could find the probability of grabbing a red in one try. They have that same probability every time they grab from the bag, so they just need to multiply the probability times the total number of tries.

A score of four indicated that students could do more than explain their own procedure. They could relate their procedure back to different parts of the problem and could often explain why they got a certain number or conducted a certain procedure. Responses at this level tended to be more culturally responsive than a score of three. Common examples of a score of four on various items within the probability and statistics assessment include:

I would start with the word problem and ask them what they are looking for and to write the word problem out in symbols. So they are looking for X glasses that are 3 times greater than Y, so Y = 3X and then I would have them figure out the number of sunglasses and find which ones fit the equation.

Show them on a spinner how the spinner is 3/5ths (60%) odd and thus there is a 60% chance each time that the spinner will land on an odd number, so 60% times 240 is 144.

Analysis across pedagogical understanding of all items on the probability and statistics assessment revealed several trends of interest. The first had to do with first demonstrating an item on a smaller scale before doing the actual problem. This is often a recommended teaching strategy for initial development of a concept. Common responses include:
I would demonstrate the problem with a smaller number and teach the process of the problem and then let them do it on their own with the bigger numbers.

Scale the problem down to smaller numbers. Ex: 3 shoes 2 pants 2 shirts: how many combos are there?

Another theme was the use of hands-on material in order to conduct experiments. The participants wanted their students to conduct experiments in order to have a better understanding of what the problem was asking for. This use of hands-on material and various methods of instruction can help to promote cultural responsiveness in the classroom. It was noted that few of the responses addressed theoretical probability versus experimental probability. This is an important concept to convey to students because the experiments conducted in the classroom may or may not be close to the theoretical probability of that experiment. Common examples include:

I would use hands on objects such as the miniature blocks that you can also put on the overhead. A lot of teachers use this to teach multiplication. Easily you could show the relationship in smaller number of sales and show what “three times greater” means. By putting each row into block groups you could add three more groups of “sunglass” blocks and see which one = which original groups sales. Then you would know that Yellow = Brown x 3

I would simply model this by using a deck of cards and discussing black and red cards. What is the probability of drawing a red card? Then you could make it more difficult and really test their understanding by discussing hearts, spades, diamonds, and other things like numbers. This would give the children an understanding of how probability works but also a real life example that they might use one day. Also it really helps them to understand a concept, but have fun with it.

If the pedagogical understanding response did not include the use of hands-on material, it generally contained some sort of representation of the item being addressed. The majority of representations presented were pictorial. Common examples of the use of pictorial representations on the understanding part of the probability and statistics assessment include:

I would explain this by drawing out in a picture format the probability of each cookie draw. Since we are only drawing out the cookie jar twice in this situation I would only need to draw to that point. The first draw is whatever the amount of cookies of that type are out of the total number of cookies. I then would draw three more branches and here I would notice that we no longer have 24 cookies; we took one away so now we have 23. Well that changes the oatmeal’s probability because we drew an oatmeal. So it would be 7/23 for example. Now the second draw on a chocolate chip doesn’t change because we didn’t draw a chocolate chip. So the probability for that is
12/23. Now we multiply across because these two probabilities multiplied together form the total probability of those two draws out of the cookie jar.

I would draw a blank for the three spots on the tag. Then I would say how many possible letters could be the first letter and that would be 21. Then I would say that 5 possible letters could be the second letter and 21 could be the last letter. Then I might begin to make a tree diagram. I would not create the whole tree but I would show them that say the first letter was B then the second letter could be A, E, I, O, or U so for all 21 consonants there would be 5 different combinations of vowels just for that first consonant. So each first consonant is taken 5 times and that gives you 21 times 5 but then there is a third consonant so now those 105 combinations are taken 21 times which gives you 21*5*21 and that equals 2205.

The three parts (content knowledge, content understanding, and pedagogical understanding) make up mathematics knowledge for teaching probability and statistics in the middle grades. Total scores for this assessment ranged from 18 to 73, a range of 55. The probability and statistics assessment had a total of 91 possible points.

Impact of Various Types of Courses

Current enrollment in MASC 450 (Integrated Mathematics) had a statistically significant contribution to mathematics knowledge for teaching number and operations, mathematics knowledge for teaching algebra, and mathematics knowledge for teaching geometry. MASC 450 (Integrated Mathematics) is a special middle-grade mathematics course developed to help students bridge their mathematics knowledge to mathematics pedagogy. It is encouraging to see this course contribute to mathematics knowledge for teaching for three of the four content areas. Current enrollment in MATH 365 (Structure of Mathematics I) had a statistically significant contribution to mathematics knowledge for teaching algebra and to mathematics knowledge for teaching geometry. MATH 365 (Structure of Mathematics I) is a course designed for elementary and middle-grade teachers that concentrates on advanced number and operations and algebra concepts. It was interesting that current enrollment in MATH 365 (Structure of Mathematics I) contributed to mathematics knowledge for teaching geometry. Perhaps this course built a good foundation for the information asked of the participants in the geometry assessment. Current enrollment in MEFB 497 (Supervised Student Teaching) had a statistically significant contribution to mathematics knowledge for teaching algebra. Students enrolled in MEFB 497 the semester of the study were student teaching. A reason for this significant contribution could be the topics student teachers experienced during their student teaching experience; perhaps there was more of a concentration on algebraic concepts during the second semester.

The interaction of current enrollment in MATH 368 (Introduction to Abstract Mathematical Structures) x MATH 403 (Mathematics and Technology) x MASC
351 (Problem Solving) x MASC 450 (Integrated Mathematics) had a statistically significant contribution to mathematics knowledge for teaching probability and statistics. These courses were chosen together because a junior student is likely to take all of them in one semester or within a semester of each other. These courses demand higher levels of thinking than previous courses and force the students to use their problem-solving abilities to be successful in the courses. Probability and statistics often requires many higher-order thinking skills to correctly predict the experiment or actually conduct the experiment. Effect sizes were small for all of the above-mentioned interactions.

MATH 403 (Math and Technology), which had already been taken by all participants, had a statistically significant contribution to mathematics knowledge for teaching algebra. MATH 403 is a mathematics and technology course geared to middle-grade teachers. Much of the technology utilized in this course used algebra and geometry concepts for examples, which could explain its contribution to mathematics knowledge for teaching algebra. The interaction of MASC 351 x MASC 450 had a statistically significant contribution to mathematics knowledge for teaching probability and statistics. Both courses focus on integrating mathematics knowledge with mathematics pedagogy and promote higher-order thinking skills. Higher-order thinking skills are essential in probability and statistics; therefore, it is encouraging to see the interaction of these two courses, which were already taken at the time of the assessment, contribute to participants’ mathematics knowledge for teaching probability and statistics. The interaction of all courses that had already been taken had a statistically significant contribution to mathematics knowledge for teaching probability and statistics. Essentially, this shows that the specified middle-grade content courses, once taken, can significantly contribute to mathematics knowledge for teaching, specifically, mathematics knowledge for teaching probability and statistics. This is definitely encouraging since this is the goal of the current middle-grade program.

CONCLUSIONS AND RECOMMENDATIONS

Although the scores reported for the mathematics knowledge for teaching number and operations could initially be discouraging to the reader, it is important not to associate letter grades with these numbers. The mean was used to calculate the averages, and it is important to remember the students were at various places in their programs. Since this study is the first of its kind, no comparisons can be made. Instead, these numbers are setting the bar for future studies. Although average mathematics knowledge for teaching scores were low among individuals, and there were several indications of student misunderstandings and misinterpretations, there was a general indication of increasing mathematics knowledge for teaching across enrollment characteristics. Again, the purpose of this study was diagnostic and to investigate growth, not to establish benchmarks or norms. It was noted, however, that there was a noticeable dip in mathematics knowledge for teaching scores during the participants’ junior semesters of courses. This could be due to a number of factors including type of courses and course loads during the junior year. Hill,
Rowan, and Ball (2005) found that teachers’ mathematics preparation positively predicted student gains in the 3rd grade. Although their results were not statistically significant, there was enough evidence to note this positive prediction. MASC 450 (Integrated Mathematics) and MEFB 460 (Math Methods in the Middle Grades) were two courses found to contribute the most to mathematics knowledge for teaching algebra, geometry, number and operations, and probability and statistics. MASC 450 (Integrated Mathematics) was a capstone course designed to help bridge the gap between mathematics content and mathematics pedagogy before sending students out into their student teaching placements.

Several other studies have found content understanding and pedagogical understanding to be weak among teachers. In the late 1980s, the National Center for Research on Teacher Education found elementary and secondary teachers were unable to explain their reasoning or why the algorithms they used worked (RAND, 2003). Instead, they exhibited a rule-bound sense of understanding. This rule-bound sense of understanding reflects the nature of teaching and the curriculum teachers experienced in elementary and secondary schools (RAND, 2003). Although many of the studies (e.g., Eisenhardt, Borko, & Underhill, 1993; Even, 1990; Graeber & Tirosh, 1991; Ma, 1999; Simon, 1993; Wheeler & Feghali, 1993) revealed “right answers” from their participants, the participants lacked an understanding of the meanings behind their procedures or solutions. Although the present study was conducted at one site focusing on one middle-grade mathematics specialist program, these results should be able to be generalized by other programs that follow a similar model.

The need for further research concerning mathematics knowledge for teaching is more essential than ever. Preliminary research has been conducted and published at the elementary grade level; however, there is a lack of literature and research concerning mathematics knowledge for teaching at the middle-grade level. This study is only a preliminary study that highlights the beginnings and ends of preservice middle-grade teachers’ mathematics knowledge for teaching. This research serves as a foundation for future studies.

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PRESERVICE TEACHERS KNOWLEDGE


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